Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2014) 38: 454 – 461

doi:10.3906/mat-1207-8

On Biharmonic Legendre curves in $\mathcal{S}$-space forms

Cihan ÖZGÜR*, **Şaban GÜVENÇ**
Department of Mathematics, Balıkesir University, Çağış, Balıkesir, Turkey

Received: 06.07.2012 • **Accepted:** 28.11.2012 • **Published Online:** 14.03.2014 • **Printed:** 11.04.2014

Abstract: We study biharmonic Legendre curves in $\mathcal{S}-$space forms. We find curvature characterizations of these special curves in 4 cases.

Key words: $\mathcal{S}-$space form, Legendre curve, biharmonic curve, Frenet curve

1. Introduction

Let (M, g) and (N, h) be 2 Riemannian manifolds and $f : (M, g) \rightarrow (N, h)$ a smooth map. The *energy functional* of f is defined by

$$E(f) = \frac{1}{2} \int_M |df|^2 \, v_g.$$

If f is a critical point of the energy functional $E(f)$, then it is called *harmonic* [10]. f is called a *biharmonic map* if it is a critical point of the bienergy functional

$$E_2(f) = \frac{1}{2} \int_M |\tau(f)|^2 \, v_g,$$

where $\tau(f)$ is *the first tension field* of f, which is defined by $\tau(f) = trace\nabla df$. The *Euler-Lagrange equation* of bienergy functional $E_2(f)$ gives the biharmonic map equation [16]

$$\tau_2(f) = -J^f(\tau(f)) = -\Delta\tau(f) - traceR^N(df, \tau(f))df = 0,$$

where J^f is the Jacobi operator of f. It is trivial that any harmonic map is biharmonic. If the map is a nonharmonic biharmonic map, then we call it *proper biharmonic*. Biharmonic submanifolds have been studied by many geometers. For example, see [2], [3], [7], [8], [11], [12], [13], [14], [15], [18], [20], [21], [22], and the references therein. In a different setting, in [9], Chen defined a biharmonic submanifold $M \subset \mathbb{E}^n$ of the Euclidean space as its mean curvature vector field H satisfies $\Delta H = 0$, where Δ is the Laplacian.

In [12] and [14], Fetcu and Oniciuc studied biharmonic Legendre curves in Sasakian space forms. As a generalization of their studies, in the present paper, we study biharmonic Legendre curves in $\mathcal{S}-$space forms. We obtain curvature characterizations of these kinds of curves.

The paper is organized as follows: In Section 2, we give a brief introduction about $\mathcal{S}-$space forms. In Section 3, we give the main results of the study.

*Correspondence: cozgur@balikesir.edu.tr
2010 *AMS Mathematics Subject Classification:* 53C25, 53C40, 53A04.

2. $\mathcal{S}-$space forms and their submanifolds

Let (M,g) be a $(2m+s)$-dimensional *framed metric manifold* [24] with a *framed metric structure* $(f,\xi_\alpha,\eta^\alpha,g)$, $\alpha\in\{1,...,s\}$, that is, f is a $(1,1)$ tensor field defining an f-*structure of rank* $2m$; $\xi_1,...,\xi_s$ are vector fields; $\eta^1,...,\eta^s$ are 1-forms; and g is a Riemannian metric on M such that for all $X,Y\in TM$ and $\alpha,\beta\in\{1,...,s\}$,

$$f^2=-I+\sum_{\alpha=1}^{s}\eta^\alpha\otimes\xi_\alpha,\quad \eta^\alpha(\xi_\beta)=\delta^\alpha_\beta,\quad f(\xi_\alpha)=0,\quad \eta^\alpha\circ f=0, \tag{2.1}$$

$$g(fX,fY)=g(X,Y)-\sum_{\alpha=1}^{s}\eta^\alpha(X)\eta^\alpha(Y), \tag{2.2}$$

$$d\eta^\alpha(X,Y)=g(X,fY)=-d\eta^\alpha(Y,X),\quad \eta^\alpha(X)=g(X,\xi). \tag{2.3}$$

$(M^{2m+s},f,\xi_\alpha,\eta^\alpha,g)$ is also called a *framed* f-*manifold* [19] or *almost* r-*contact metric manifold* [23]. If the Nijenhuis tensor of f equals $-2d\eta^\alpha\otimes\xi_\alpha$ for all $\alpha\in\{1,...,s\}$, then $(f,\xi_\alpha,\eta^\alpha,g)$ is called $\mathcal{S}$-*structure* [4].

If $s=1$, a framed metric structure is an almost contact metric structure and an $\mathcal{S}$-structure is a Sasakian structure. If a framed metric structure on M is an $\mathcal{S}$-structure, then the following equations hold [4]:

$$(\nabla_X f)Y=\sum_{\alpha=1}^{s}\left\{g(fX,fY)\xi_\alpha-\eta^\alpha(Y)f^2X\right\}, \tag{2.4}$$

$$\nabla\xi_\alpha=-f,\ \alpha\in\{1,...,s\}. \tag{2.5}$$

In the case of Sasakian structure ($s=1$), (2.5) can be calculated using (2.4) .

A *plane section* in T_pM is an f-section if there exists a vector $X\in T_pM$ orthogonal to $\xi_1,...,\xi_s$ such that $\{X,fX\}$ span the section. The sectional curvature of an f-*section* is called an f-*sectional curvature*. In an $\mathcal{S}$-manifold of constant f-sectional curvature, the *curvature tensor* R *of* M is of the form

$$\begin{aligned} R(X,Y)Z=&\sum_{\alpha,\beta}\{\eta^\alpha(X)\eta^\beta(Z)f^2Y-\eta^\alpha(Y)\eta^\beta(Z)f^2X\\ &-g(fX,fZ)\eta^\alpha(Y)\xi_\beta+g(fY,fZ)\eta^\alpha(X)\xi_\beta\}\\ &+\tfrac{c+3s}{4}\left\{-g(fY,fZ)f^2X+g(fX,fZ)f^2Y\right\}\\ &\tfrac{c-s}{4}\left\{g(X,fZ)fY-g(Y,fZ)fX+2g(X,fY)fZ\right\}, \end{aligned} \tag{2.6}$$

for all $X,Y,Z\in TM$ [6]. An $\mathcal{S}$-manifold of constant f-sectional curvature c is called an $\mathcal{S}$-*space form*, which is denoted by $M(c)$. When $s=1$, an $\mathcal{S}$-space form becomes a Sasakian space form [5].

A submanifold of an $\mathcal{S}$-manifold is called an *integral submanifold* if $\eta^\alpha(X)=0,\ \alpha=1,...,s,$ for every tangent vector X [17]. We call a 1-dimensional integral submanifold of an $\mathcal{S}$-space form $(M^{2m+s},f,\xi_\alpha,\eta^\alpha,g)$ a *Legendre curve of* M. In other words, a curve $\gamma:I\to M=(M^{2m+s},f,\xi_\alpha,\eta^\alpha,g)$ is called a Legendre curve if $\eta^\alpha(T)=0,$ for every $\alpha=1,...s,$ where T is the tangent vector field of γ.

3. Biharmonic Legendre curves in $\mathcal{S}$-space forms

Let $\gamma : I \to M$ be a curve parametrized by arc length in an n-dimensional Riemannian manifold (M, g). If there exists orthonormal vector fields $E_1, E_2, ..., E_r$ along γ such that

$$\begin{aligned} E_1 &= \gamma' = T, \\ \nabla_T E_1 &= \kappa_1 E_2, \\ \nabla_T E_2 &= -\kappa_1 E_1 + \kappa_2 E_3, \\ &\dots \\ \nabla_T E_r &= -\kappa_{r-1} E_{r-1}, \end{aligned} \tag{3.7}$$

then γ is called a *Frenet curve of osculating order* r, where $\kappa_1, ..., \kappa_{r-1}$ are positive functions on I and $1 \leq r \leq n$.

A Frenet curve of osculating order 1 is a *geodesic*; a Frenet curve of osculating order 2 is called a *circle* if κ_1 is a nonzero positive constant; a Frenet curve of osculating order $r \geq 3$ is called a *helix of order* r if $\kappa_1, ..., \kappa_{r-1}$ are nonzero positive constants; a helix of order 3 is shortly called a *helix*.

Now let $(M^{2m+s}, f, \xi_\alpha, \eta^\alpha, g)$ be an $\mathcal{S}$-space form and $\gamma : I \to M$ a Legendre Frenet curve of osculating order r. Differentiating

$$\eta^\alpha(T) = 0 \tag{3.8}$$

and using (3.7), we find

$$\eta^\alpha(E_2) = 0, \ \alpha \in \{1, ..., s\}. \tag{3.9}$$

By the use of (2.1), (2.2), (2.3), (2.6), (3.7), and (3.9), it can be seen that

$$\nabla_T \nabla_T T = -\kappa_1^2 E_1 + \kappa_1' E_2 + \kappa_1 \kappa_2 E_3,$$

$$\begin{aligned} \nabla_T \nabla_T \nabla_T T &= -3\kappa_1 \kappa_1' E_1 + \left(\kappa_1'' - \kappa_1^3 - \kappa_1 \kappa_2^2\right) E_2 \\ &\quad + \left(2\kappa_1' \kappa_2 + \kappa_1 \kappa_2'\right) E_3 + \kappa_1 \kappa_2 \kappa_3 E_4, \end{aligned}$$

$$R(T, \nabla_T T)T = -\kappa_1 \frac{(c+3s)}{4} E_2 - 3\kappa_1 \frac{(c-s)}{4} g(fT, E_2) fT.$$

Thus, we have

$$\begin{aligned} \tau_2(\gamma) &= \nabla_T \nabla_T \nabla_T T - R(T, \nabla_T T)T \\ &= -3\kappa_1 \kappa_1' E_1 \\ &\quad + \left(\kappa_1'' - \kappa_1^3 - \kappa_1 \kappa_2^2 + \kappa_1 \frac{(c+3s)}{4}\right) E_2 \\ &\quad + (2\kappa_1' \kappa_2 + \kappa_1 \kappa_2') E_3 + \kappa_1 \kappa_2 \kappa_3 E_4 \\ &\quad + 3\kappa_1 \frac{(c-s)}{4} g(fT, E_2) fT. \end{aligned} \tag{3.10}$$

Let $k = \min\{r, 4\}$. From (3.10), the curve γ is proper biharmonic if and only if $\kappa_1 > 0$ and

(1) $c = s$ or $fT \perp E_2$ or $fT \in span\{E_2, ..., E_k\}$; and

(2) $g(\tau(\gamma), E_i) = 0$, for any $i = \overline{1, k}$.

We can therefore state the following theorem:

Theorem 3.1 *Let γ be a Legendre Frenet curve of osculating order r in an $\mathcal{S}$-space form $(M^{2m+s}, f, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1,...,s\}$, and $k = \min\{r, 4\}$. Then γ is proper biharmonic if and only if*

(1) *$c = s$ or $fT \perp E_2$ or $fT \in span\{E_2, ..., E_k\}$; and*

(2) *the first k of the following equations are satisfied (replacing $\kappa_k = 0$):*

$$\begin{gathered} \kappa_1 = constant > 0, \\ \kappa_1^2 + \kappa_2^2 = \frac{c+3s}{4} + \frac{3(c-s)}{4}\left[g(fT, E_2)\right]^2, \\ \kappa_2' + \frac{3(c-s)}{4} g(fT, E_2) g(fT, E_3) = 0, \\ \kappa_2\kappa_3 + \frac{3(c-s)}{4} g(fT, E_2) g(fT, E_4) = 0. \end{gathered}$$

Now we give the interpretations of Theorem 3.1.

Case I. $c = s$.
In this case γ is proper biharmonic if and only if

$$\begin{gathered} \kappa_1 = \text{constant} > 0, \\ \kappa_1^2 + \kappa_2^2 = s, \\ \kappa_2 = \text{constant}, \\ \kappa_2\kappa_3 = 0. \end{gathered}$$

Theorem 3.2 *Let γ be a Legendre Frenet curve in an $\mathcal{S}$-space form $(M^{2m+s}, f, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1,...,s\}$, $c = s$, and $(2m+s) > 3$. Then γ is proper biharmonic if and only if either γ is a circle with $\kappa_1 = \sqrt{s}$ or a helix with $\kappa_1^2 + \kappa_2^2 = s$.*

Remark 3.1 *If $2m+s = 3$, then $m = s = 1$. So M is a 3-dimensional Sasakian space form. Since a Legendre curve in a Sasakian 3-manifold has torsion 1 (see [1]), we can write $\kappa_1 > 0$ and $\kappa_2 = 1$, which contradicts $\kappa_1^2 + \kappa_2^2 = s = 1$. Hence, γ cannot be proper biharmonic.*

Case II. $c \neq s$, $fT \perp E_2$.
In this case, $g(fT, E_2) = 0$. From Theorem 3.1, we obtain

$$\begin{gathered} \kappa_1 = \text{constant} > 0, \\ \kappa_1^2 + \kappa_2^2 = \frac{c+3s}{4}, \\ \kappa_2 = \text{constant}, \\ \kappa_2\kappa_3 = 0. \end{gathered} \tag{3.11}$$

First, we give the following proposition:

Proposition 3.1 *Let γ be a Legendre Frenet curve of osculating order 3 in an $\mathcal{S}$-space form $(M^{2m+s}, f, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1,...,s\}$, and $fT \perp E_2$. Then $\{T = E_1, E_2, E_3, fT, \nabla_T fT, \xi_1, ..., \xi_s\}$ is linearly independent at any point of γ. Therefore, $m \geq 3$.*

Proof Since γ is a Frenet curve of osculating order 3, we can write

$$\begin{aligned} E_1 &= \gamma' = T, \\ \nabla_T E_1 &= \kappa_1 E_2, \\ \nabla_T E_2 &= -\kappa_1 E_1 + \kappa_2 E_3, \\ \nabla_T E_3 &= -\kappa_2 E_2. \end{aligned} \tag{3.12}$$

The system

$$S_1 = \{T = E_1, E_2, E_3, fT, \nabla_T fT, \xi_1, ..., \xi_s\}$$

has only nonzero vectors. Using (2.1), (2.2), (2.3), and (2.4), we find

$$\nabla_T fT = \sum_{\alpha=1}^{s} \xi_\alpha + \kappa_1 f E_2. \tag{3.13}$$

So by the use of (3.8), (3.9), (3.12), and (3.13), we have

$$\begin{aligned} T &\perp E_2, T \perp E_3,\ T \perp E_4,\ T \perp fT, \\ T &\perp \nabla_T fT,\ T \perp \xi_\alpha \text{ for all } \alpha \in \{1, ..., s\}. \end{aligned}$$

Hence, S_1 is linearly independent if and only if $S_2 = \{E_2, E_3, fT, \nabla_T fT, \xi_1, ..., \xi_s\}$ is linearly independent. From the assumption we have $E_2 \perp fT$. From (3.9), $E_2 \perp \xi_\alpha$ for all $\alpha \in \{1, ..., s\}$. Using (2.3), (3.12), and (3.13), we have $E_2 \perp E_3$ and $E_2 \perp \nabla_T fT$. So S_2 is linearly independent if and only if $S_3 = \{E_3, fT, \nabla_T fT, \xi_1, ..., \xi_s\}$ is linearly independent. Differentiating $g(fT, E_2) = 0$ and using (3.12) and (3.13), we find $g(fT, E_3) = 0$. Hence, $fT \perp E_3$. Using (2.1) and (2.3), we find $g(fT, \xi_\alpha) = 0$, that is, $fT \perp \xi_\alpha$ for all $\alpha \in \{1, ..., s\}$. Using (2.2) and (3.13), we obtain $g(fT, \nabla_T fT) = 0$. So S_3 is linearly independent if and only if $S_4 = \{E_3, \nabla_T fT, \xi_1, ..., \xi_s\}$ is linearly independent. Differentiating $\eta^\alpha(E_2) = 0$, we have $\eta^\alpha(E_3) = 0$, $\alpha \in \{1, ..., s\}$. Thus $E_3 \perp \xi_\alpha$ for all $\alpha \in \{1, ..., s\}$. If we differentiate $g(fT, E_3) = 0$, we get $g(\nabla_T fT, E_3) = 0$, that is, $E_3 \perp \nabla_T fT$. So S_4 is linearly independent if and only if $S_5 = \{\nabla_T fT, \xi_1, ..., \xi_s\}$ is linearly independent. Since $\kappa_1 \neq 0$ and $fE_2 \perp \xi_\alpha$ for all $\alpha \in \{1, ..., s\}$, equation (3.13) gives us $\nabla_T fT \notin span\{\xi_1, ..., \xi_s\}$. So S_5 is linearly independent.

Since $\{T = E_1, E_2, E_3, fT, \nabla_T fT, \xi_1, ..., \xi_s\}$ is linearly independent, $\dim M = 2m + s \geq s + 5$. Hence, $m \geq 3$. □

Now we can state the following Theorem:

Theorem 3.3 *Let γ be a Legendre Frenet curve in an $\mathcal{S}$-space form $(M^{2m+s}, f, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1, ..., s\}$, $c \neq s$, and $fT \perp E_2$. Then γ is proper biharmonic if and only if either*

(1) $m \geq 2$ and γ is a circle with $\kappa_1 = \frac{1}{2}\sqrt{c + 3s}$, where $c > -3s$ and $\{T = E_1, E_2, fT, \nabla_T fT, \xi_1, ..., \xi_s\}$ is linearly independent; or

(2) $m \geq 3$ and γ is a helix with $\kappa_1^2 + \kappa_2^2 = \frac{c+3s}{4}$, where $c > -3s$ and $\{T = E_1, E_2, E_3, fT, \nabla_T fT, \xi_1, ..., \xi_s\}$ is linearly independent.

If $c \leq -3s$, then γ is biharmonic if and only if it is a geodesic.

Case III. $c \neq s$, $fT \parallel E_2$.

In this case, $fT = \pm E_2, g(fT, E_2) = \pm 1$, $g(fT, E_3) = g(\pm E_2, E_3) = 0$, and $g(fT, E_4) = g(\pm E_2, E_4) = 0$. From Theorem 3.1, γ is biharmonic if and only if

$$\begin{aligned} \kappa_1 &= \text{constant} > 0, \\ \kappa_1^2 + \kappa_2^2 &= c, \\ \kappa_2 &= \text{constant}, \\ \kappa_2 \kappa_3 &= 0. \end{aligned}$$

We can assume that $fT = E_2$. From equation (2.1), we get

$$fE_2 = f^2T = -T + \sum_{\alpha=1}^{s} \eta^{\alpha}(T)\xi_{\alpha} = -T. \tag{3.14}$$

From (3.13) and (3.14), we find

$$\nabla_T fT = \sum_{\alpha=1}^{s} \xi_{\alpha} - \kappa_1 T. \tag{3.15}$$

Using (3.7) and (3.15), we can write

$$\kappa_2 E_3 = \sum_{\alpha=1}^{s} \xi_{\alpha},$$

which gives us

$$\begin{aligned} \kappa_2 &= \left\| \sum_{\alpha=1}^{s} \xi_{\alpha} \right\| = \sqrt{s}, \\ E_3 &= \frac{1}{\sqrt{s}} \sum_{\alpha=1}^{s} \xi_{\alpha}, \\ \eta^{\alpha}(E_3) &= \frac{1}{\sqrt{s}},\ \alpha \in \{1, ..., s\}. \end{aligned}$$

Thus by the use of Theorem 3.1, we have the following Theorem:

Theorem 3.4 *Let γ be a Legendre Frenet curve in an $\mathcal{S}$-space form $(M^{2m+s}, f, \xi_{\alpha}, \eta^{\alpha}, g)$, $\alpha \in \{1, ..., s\}$, $c \neq s$, and $fT \parallel E_2$. Then*

$$\left\{ T, fT, \frac{1}{\sqrt{s}} \sum_{\alpha=1}^{s} \xi_{\alpha} \right\}$$

is the Frenet frame field of γ and γ is proper biharmonic if and only if it is a helix with $\kappa_1 = \sqrt{c-s}$ and $\kappa_2 = \sqrt{s}$, where $c > s$. If $c \leq s$, then γ is biharmonic if and only if it is a geodesic.

Case IV. $c \neq s$ and $g(fT, E_2)$ is not constant 0, 1, or -1.

Now, let $(M^{2m+s}, f, \xi_{\alpha}, \eta^{\alpha}, g)$ be an $\mathcal{S}$-space form, $\alpha \in \{1, ..., s\}$, and $\gamma : I \to M$ a Legendre curve of osculating order r, where $4 \leq r \leq 2m + s$ and $m \geq 2$. If γ is biharmonic, then $fT \in span\{E_2, E_3, E_4\}$. Let $\theta(t)$ denote the angle function between fT and E_2, that is, $g(fT, E_2) = \cos\theta(t)$. Differentiating $g(fT, E_2)$ along γ and using (2.1), (2.3), (3.7), and (3.13), we find

$$\begin{aligned} -\theta'(t)\sin\theta(t) &= \nabla_T g(fT, E_2) = g(\nabla_T fT, E_2) + g(fT, \nabla_T E_2) \\ &= g(\sum_{\alpha=1}^{s} \xi_{\alpha} + \kappa_1 fE_2, E_2) + g(fT, -\kappa_1 T + \kappa_2 E_3) \\ &= \kappa_2 g(fT, E_3). \end{aligned} \tag{3.16}$$

If we write $fT = g(fT, E_2)E_2 + g(fT, E_3)E_3 + g(fT, E_4)E_4$, Theorem 3.1 gives us

$$\begin{gathered}\kappa_1 = constant > 0, \\ \kappa_1^2 + \kappa_2^2 = \tfrac{c+3s}{4} + \tfrac{3(c-s)}{4}\cos^2\theta, \\ \kappa_2' + \tfrac{3(c-s)}{4}\cos\theta g(fT, E_3) = 0, \\ \kappa_2\kappa_3 + \tfrac{3(c-s)}{4}\cos\theta g(fT, E_4) = 0.\end{gathered}$$

If we multiply the third equation of the above system with $2\kappa_2$, using (3.16), we obtain

$$2\kappa_2\kappa_2' + \frac{3(c-s)}{4}(-2\theta'\cos\theta\sin\theta) = 0,$$

which is equivalent to

$$\kappa_2^2 = -\frac{3(c-s)}{4}\cos^2\theta + \omega_0, \tag{3.17}$$

where ω_0 is a constant. If we write (3.17) in the second equation, we have

$$\kappa_1^2 = \frac{c+3s}{4} + \frac{3(c-s)}{2}\cos^2\theta + \omega_0.$$

Thus, θ is a constant. From (3.16) and (3.17), we find $g(fT, E_3) = 0$ and $\kappa_2 =$constant> 0. Since $\|fT\| = 1$ and $fT = \cos\theta E_2 + g(fT, E_4)E_4$, we get $g(fT, E_4) = \sin\theta$. From the assumption $g(fT, E_2)$ is not constant 0, 1, or -1, it is clear that $\theta \in (0, 2\pi)\setminus\left\{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}$. Now we can state the following Theorem:

Theorem 3.5 *Let $\gamma : I \to M$ be a Legendre curve of osculating order r in an $\mathcal{S}$-space form $(M^{2m+s}, f, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1, ..., s\}$, where $r \geq 4$, $m \geq 2$, $c \neq s$, $g(fT, E_2)$ is not constant 0, 1, or -1. Then γ is proper biharmonic if and only if*

$$\begin{aligned}\kappa_i &= constant > 0,\ i \in \{1, 2, 3\}, \\ \kappa_1^2 + \kappa_2^2 &= \frac{1}{4}\left[c + 3s + 3(c-s)\cos^2\theta\right], \\ \kappa_2\kappa_3 &= \frac{3(s-c)\sin 2\theta}{8},\end{aligned}$$

where $c > -3s$, $fT = \cos\theta E_2 + \sin\theta E_4$, $\theta \in (0, 2\pi)\setminus\left\{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}$ is a constant such that $c+3s+3(c-s)\cos^2\theta > 0$, and $3(s-c)\sin 2\theta > 0$. If $c \leq -3s$, then γ is biharmonic if and only if it is a geodesic.

Acknowledgments

The authors are thankful to the referee for his/her valuable comments towards the improvement of the paper.

References

[1] Baikoussis, C., Blair, D.E.: On Legendre curves in contact 3-manifolds. Geom. Dedicata 49, 135–142 (1994).

[2] Balmuş, A. Montaldo, S., Oniciuc, C.: Classification results for biharmonic submanifolds in spheres. Israel J. Math. 168, 201–220 (2008).

[3] Balmuş, A. Montaldo, S., Oniciuc, C.: Biharmonic hypersurfaces in 4-dimensional space forms. Math. Nachr. 283, 1696–1705 (2010).

[4] Blair, D.E.: Geometry of manifolds with structural group $\mathcal{U}(n)\times\mathcal{O}(s)$. J. Differential Geometry 4, 155–167 (1970).

[5] Blair, D.E.: Riemannian Geometry of Contact and Symplectic Manifolds. (Boston. Birkhauser 2002).

[6] Cabrerizo, J.L., Fernandez, L.M., Fernandez, M.: The curvature of submanifolds of an S-space form. Acta Math. Hungar. 62, 373–383 (1993).

[7] Caddeo, R., Montaldo, S., Oniciuc, C.: Biharmonic submanifolds of S^3. Internat. J. Math. 12, 867–876 (2001).

[8] Caddeo, R., Montaldo, S., Oniciuc, C.: Biharmonic submanifolds in spheres. Israel J. Math. 130, 109–123 (2002).

[9] Chen, B.Y.: A report on submanifolds of finite type. Soochow J. Math. 22, 117–337 (1996).

[10] Eells, J. Jr, Sampson, J.H.: Harmonic mappings of Riemannian manifolds. Amer. J. Math. 86, 109–160 (1964).

[11] Fetcu, D.: Biharmonic curves in the generalized Heisenberg group. Beitrage zur Algebra und Geometrie 46, 513–521 (2005).

[12] Fetcu, D.: Biharmonic Legendre curves in Sasakian space forms. J. Korean Math. Soc. 45, 393–404 (2008).

[13] Fetcu, D., Oniciuc, C.: Biharmonic hypersurfaces in Sasakian space forms. Differential Geom. Appl. 27, 713–722 (2009).

[14] Fetcu, D., Oniciuc, C.: Explicit formulas for biharmonic submanifolds in Sasakian space forms. Pacific J. Math. 240, 85–107 (2009).

[15] Fetcu, D., Loubeau, E., Montaldo, S., Oniciuc, C.: Biharmonic submanifolds of $\mathbb{CP}^n$. Math. Z. 266, 505–531 (2010).

[16] Jiang, G.Y.: 2-Harmonic maps and their first and second variational formulas. Chinese Ann. Math. Ser. A 7, 389–402 (1986).

[17] Kim, J.S., Dwivedi, M.K., Tripathi, M.M.: Ricci curvature of integral submanifolds of an $\mathcal{S}$-space form. Bull. Korean Math. Soc. 44 , 395–406 (2007).

[18] Montaldo, S., Oniciuc, C.: A short survey on biharmonic maps between Riemannian manifolds. Rev. Un. Mat. Argentina 47, 1–22 (2006).

[19] Nakagawa, H.: On framed f-manifolds. Kodai Math. Sem. Rep. 18, 293–306 (1966).

[20] Ou, Y.L.: p-Harmonic morphisms, biharmonic morphisms,and nonharmonic biharmonic maps. J. Geom. Phys. 56, 358–374 (2006).

[21] Ou, Y.L.: Biharmonic hypersurfaces in Riemannian manifolds. Pacific J. Math. 248, 217–232 (2010)

[22] Ou, Y.L.: Some constructions of biharmonic maps and Chen's conjecture on biharmonic hypersurfaces. J. Geom. Phys. 62, 751–762 (2012).

[23] Vanzura, J.: Almost r-contact structures. Ann. Scuola Norm. Sup. Pisa. 26, 97–115 (1972).

[24] Yano, K., Kon, M.: Structures on Manifolds. Series in Pure Mathematics, 3. (Singapore. World Scientific Publishing Co. 1984).