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# Improved direct and converse theorems in weighted Lorentz spaces

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## Abstract

In the present work we prove the equivalence of the fractional modulus of smoothness to the realization functional and to the Peetre  $K$ -functional in weighted Lorentz spaces. Using this equivalence we obtain an improvement of the direct approximation theorem. Furthermore we prove the improved converse theorem in this space.

## 1 Introduction and main results

In approximation theory improvements of direct and inverse theorems have been investigated by several authors in different function spaces [1, 9, 12, 18, 20, 21]. In this paper we deal with the improved direct and inverse approximation theorems in the weighted Lorentz space  $L_{\omega}^{pq}(\mathbb{T})$  with Muckenhoupt weights. To obtain the improved direct theorem we need the realization and characterization theorem in  $L_{\omega}^{pq}(\mathbb{T})$ . Therefore we will prove a realization result and an equivalence relation between the modulus of smoothness and the Peetre  $K$ -functional in  $L_{\omega}^{pq}(\mathbb{T})$ . Furthermore, the realization result has a lot of applications [6]. In particular, it is used to get Ul'yanov type inequalities [8]. First, we give some definitions and properties.

Let  $\mathbb{T} := [-\pi, \pi]$  and  $\omega : \mathbb{T} \rightarrow [0, \infty]$  be a weight function i.e., an almost everywhere positive measurable function. We define the decreasing rearrange-

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ment  $f_\omega^*(t)$  [11] of  $f : \mathbb{T} \rightarrow \mathbb{R}$  with respect to the Borel measure

$$\omega(e) = \int_e \omega(x) dx,$$

by

$$f_\omega^*(t) = \inf \{ \tau \geq 0 : \omega(x \in \mathbb{T} : |f(x)| > \tau) \leq t \}.$$

The weighted Lorentz space  $L_\omega^{pq}(\mathbb{T})$  is defined [11] as

$$L_\omega^{pq}(\mathbb{T}) = \{ f \in \mathbf{M}(\mathbb{T}) : \|f\|_{pq,\omega} = \left( \int_{\mathbb{T}} (f^{**}(t))^q t^{\frac{q}{p}} \frac{dt}{t} \right)^{1/q} < \infty, 1 < p, q < \infty \},$$

where  $\mathbf{M}(\mathbb{T})$  is the set of  $2\pi$  periodic integrable functions on  $\mathbb{T}$  and

$$f^{**}(t) = \frac{1}{t} \int_0^t f_\omega^*(u) du.$$

If  $p = q$ ,  $L_\omega^{pq}(\mathbb{T})$  turns into the weighted Lebesgue space  $L_\omega^p(\mathbb{T})$  [11, p.20].

A weight function  $\omega : \mathbb{T} \rightarrow [0, \infty]$  belongs to the Muckenhoupt class  $A_p$  [17],  $1 < p < \infty$ , if

$$\sup \frac{1}{|I|} \int_I \omega(x) dx \left( \frac{1}{|I|} \int_I \omega^{1-p'}(x) dx \right)^{p-1} = C_{A_p} < \infty, \quad p' := \frac{p}{p-1}$$

with a finite constant  $C_{A_p}$  independent of  $I$ , where the supremum is taken over all intervals  $I$  with length  $\leq 2\pi$  and  $|I|$  denotes the length of  $I$ . The constant  $C_{A_p}$  is called the Muckenhoupt constant of  $\omega$ .

By the proof of [14, Prop. 3.3], we know that  $L_\omega^{pq}(\mathbb{T}) \subset L^1(\mathbb{T})$ . Let

$$S[f] := \sum_{k=-\infty}^{\infty} c_k e^{ikx} \tag{1}$$

be the Fourier series of a function  $f \in L^1(\mathbb{T})$ . Assume that

$$\int_{\mathbb{T}} f(x) dx = 0. \tag{2}$$

For  $\alpha \in \mathbb{R}^+$ , we define the  $\alpha$ -th fractional integral of  $f$  as [22, v.2, p.134]

$$I_\alpha(x, f) := \sum_{k \in \mathbb{Z}^*} c_k (ik)^{-\alpha} e^{ikx},$$

with

$$(ik)^{-\alpha} := |k|^{-\alpha} e^{(-1/2)\pi i \alpha \operatorname{sign} k}$$

as principal value.

We define the fractional derivative of a function  $f \in L^1(\mathbb{T})$ , satisfying (2), as

$$f^{(\alpha)}(x) := \frac{d^{[\alpha]+1}}{dx^{[\alpha]+1}} I_{1+\alpha-[\alpha]}(x, f),$$

whenever the right hand side exists.

For a function  $f \in L_\omega^{pq}(\mathbb{T})$ ,  $1 < p, q < \infty$ ,  $w \in A_p$ , Steklov's mean operator is defined as

$$\sigma_h f(x) := \frac{1}{2h} \int_{x-h}^{x+h} f(u) du, \quad x \in \mathbb{T}.$$

Whenever  $\omega \in A_p$ ,  $1 < p, q < \infty$ , the Hardy-Littlewood maximal function of  $f \in L_\omega^{pq}(\mathbb{T})$  belongs to  $L_\omega^{pq}(\mathbb{T})$  [5, Theorem 3]. Therefore the operator  $\sigma_h f$  belongs to  $L_\omega^{pq}(\mathbb{T})$ . Using this fact and putting  $x, t \in \mathbb{T}$ ,  $r \in \mathbb{R}^+$ ,  $\omega \in A_p$  and  $f \in L_\omega^{pq}(\mathbb{T})$ ,  $1 < p, q < \infty$ , we define

$$\begin{aligned} \sigma_t^r f(x) &:= (I - \sigma_t)^r f(x) \\ &= \sum_{k=0}^{\infty} (-1)^k \binom{r}{k} \frac{1}{(2t)^k} \int_{-t}^t \cdots \int_{-t}^t f(x + u_1 + \cdots + u_k) du_1 \dots du_k, \end{aligned}$$

where  $\binom{r}{k}$  are the binomial coefficients.

Since

$$\left| \binom{\alpha}{k} \right| \leq \frac{c}{k^{\alpha+1}}, \quad k \in \mathbb{N}$$

(see [19, p.14, (1.51)]), we have

$$\sum_{k=0}^{\infty} \left| \binom{\alpha}{k} \right| < \infty,$$

and therefore

$$\|\sigma_t^\alpha f\|_{pq,\omega} \leq c \|f\|_{pq,\omega} < \infty, \tag{3}$$

for  $f \in L_\omega^{pq}(\mathbb{T})$ ,  $1 < p, q < \infty$ ,  $\omega \in A_p$ .

For  $1 < p, q < \infty$ ,  $f \in L_\omega^{pq}(\mathbb{T})$  and  $r \in \mathbb{R}^+$ , we define the fractional modulus of smoothness of index  $r$  as

$$\Omega_r(f, \delta)_{pq,\omega} := \sup_{0 < h_i, t < \delta} \left\| \prod_{i=1}^{[r]} (I - \sigma_{h_i}) (I - \sigma_t)^{r-[r]} f \right\|_{pq,\omega}, \tag{4}$$

where  $[r] := \max \{n \in \mathbb{N} : n \leq r\}$ . Since the operator  $\sigma_t$  is bounded in  $L_\omega^{pq}(\mathbb{T})$ ,  $1 < p, q < \infty$ ,  $\omega \in A_p$  we have by (3) that

$$\Omega_r(f, \delta)_{pq,\omega} \leq c \|f\|_{pq,\omega}$$

where the constant  $c > 0$  only depends on  $r, p, q$  and  $C_{A_p}$ .

*Remark 1.1.* Let  $r \in \mathbb{R}^+$ ,  $1 < p, q < \infty$ ,  $\omega \in A_p$  and  $f \in L_\omega^{pq}(\mathbb{T})$ . For  $\delta > 0$ , the modulus of smoothness  $\Omega_r(f, \delta)_{pq,\omega}$  has the following properties.

- (i)  $\Omega_r(f, \delta)_{pq,\omega}$  is sub-additive in  $f$ , and a non-negative, non-decreasing function of  $\delta \geq 0$ .
- (ii)  $\lim_{\delta \rightarrow 0} \Omega_r(f, \delta)_{pq,\omega} = 0$ .

By  $E_n(f)_{pq,\omega}$  we denote the best approximation of  $f \in L_\omega^{pq}(\mathbb{T})$  by polynomials in  $\mathcal{T}_n$ , the set of trigonometric polynomials of degree  $\leq n$ :

$$E_n(f)_{pq,\omega} = \inf_{T_n \in \mathcal{T}_n} \|f - T_n\|_{pq,\omega}.$$

In this paper we will use the following notations:

$$\begin{aligned} A(x) \approx B(x) &\Leftrightarrow \exists c_1, c_2 > 0 : c_1 B(x) \leq A(x) \leq c_2 B(x) \\ A(x) \preceq B(x) &\Leftrightarrow \exists c > 0 : A(x) \leq c B(x). \end{aligned}$$

Our main results are now the following.

**Theorem 1.2.** *If  $r \in \mathbb{R}^+$ ,  $1 < p, q < \infty$ ,  $\omega \in A_p$  and  $f \in L_\omega^{pq}(\mathbb{T})$ , then there exists a constant  $c > 0$  depending only on  $r, p, q$  and  $C_{A_p}$  such that*

$$E_n(f)_{pq,\omega} \leq c \Omega_r\left(f, \frac{1}{n+1}\right)_{pq,\omega} \tag{5}$$

holds for  $n + 1 \in \mathbb{N}$ .

The analogues of this direct approximation theorem were obtained in [10] for  $r \in \mathbb{N}$ ,  $f \in L_\omega^p(\mathbb{T})$ ,  $1 < p < \infty$ ,  $\omega \in A_p$  and in [2] for  $r \in \mathbb{N}$ ,  $f \in L_\omega^{pq}(\mathbb{T})$ ,  $1 < p, q < \infty$ ,  $\omega \in A_p$  with the modulus of smoothness

$$W_r\left(f, \frac{1}{n}\right)_{L_\omega^{pq}} := \sup_{0 \leq h_i \leq 1/n} \left\| \prod_{i=1}^r (I - \sigma_{h_i}) f \right\|_{L_\omega^{pq}},$$

and in [1] for  $r \in \mathbb{R}^+$ ,  $f \in L_\omega^p(\mathbb{T})$ ,  $1 < p < \infty$ ,  $\omega \in A_p$  with the fractional modulus of smoothness (4).

For  $f \in L_\omega^{pq}(\mathbb{T})$ ,  $t, r > 0$  and  $1 < p, q < \infty$ , the Peetre  $K$ -functional is defined as

$$K_r(f, t; L_\omega^{pq}, W_{pq,\omega}^r) := \inf_{g \in W_{pq,\omega}^r} \{ \|f - g\|_{L_\omega^{pq}} + t^r \|g^{(r)}\|_{L_\omega^{pq}} \}.$$

Here  $W_{pq,\omega}^r := \left\{ g(x) \in L_\omega^{pq} : g^{(r)} \in L_\omega^{pq} \right\}$ .

We define the realization functional for  $f \in L_\omega^{pq}(\mathbb{T})$  by

$$R_r(f, 1/n, L_\omega^{pq}) := \left\{ \|f - t_n^*\|_{L_\omega^{pq}} + \frac{1}{n^r} \|(t_n^*)^{(r)}\|_{L_\omega^{pq}} \right\},$$

for  $r > 0$ ,  $1 < p, q < \infty$ ,  $n \in \mathbb{N}$ . Here  $t_n^*$  denotes the best approximating trigonometric polynomial for  $f$ . The following theorem holds.

**Theorem 1.3.** *If  $\mathbb{R}^+$ ,  $f \in L_\omega^{pq}(\mathbb{T})$ ,  $1 < p, q < \infty$  and  $\omega \in A_p$ , then the equivalence*

$$\Omega_r\left(f, \frac{1}{n}\right)_{pq,\omega} \approx R_{2r}\left(f, 1/n, L_\omega^{pq}\right) \tag{6}$$

holds for  $n = 1, 2, 3, \dots$ . Furthermore, we have, for  $\delta \geq 0$ ,

$$\Omega_r(f, \delta)_{pq,\omega} \approx K_{2r}(f, \delta; L_\omega^{pq}, W_{pq,\omega}^r). \tag{7}$$

Here the equivalence constants only depend on  $r, p, q$  and  $C_{A_p}$ .

**Corollary 1.4.** Let  $r \in \mathbb{R}^+$ ,  $f \in L_\omega^{pq}(\mathbb{T})$ ,  $1 < p, q < \infty$ , and  $\omega \in A_p$ . Then

$$\Omega_r(f, \lambda\delta)_{pq, \omega} \preceq (1 + [\lambda])^{2r} \Omega_r(f, \delta)_{pq, \omega}, \quad \delta, \lambda > 0$$

and

$$\Omega_r(f, \delta)_{pq, \omega} \delta^{-2r} \preceq \Omega_r(f, \delta_1)_{pq, \omega} \delta_1^{-2r}, \quad 0 < \delta_1 \leq \delta.$$

An improvement of (5) is given by the following theorem.

**Theorem 1.5.** If  $r \in \mathbb{R}^+$ ,  $f \in L_\omega^{pq}(\mathbb{T})$ ,  $1 < p, q < \infty$  and  $\omega \in A_p$ , then there exists a constant  $c > 0$  depending on  $r, p, q$  and  $C_{A_p}$  such that for  $n = 1, 2, 3, \dots$

$$\left( \prod_{j=1}^n E_j(f)_{pq, \omega} \right)^{1/n} \leq c \Omega_r\left(f, \frac{1}{n}\right)_{pq, \omega}. \tag{8}$$

*Remark 1.6.* The inequality (8) is never worse than the classical Jackson inequality. Since  $E_n(f)_{pq, \omega} \rightarrow 0$  as  $n \rightarrow \infty$  we obtain that

$$E_n(f)_{pq, \omega} \leq \left( \prod_{j=1}^n E_j(f)_{pq, \omega} \right)^{1/n} \leq c \Omega_r\left(f, \frac{1}{n}\right)_{pq, \omega}.$$

On the other hand, in some cases the inequality (8) gives better results than the classical Jackson inequality. For example, if  $E_n(f)_{pq, \omega} = 2^{-n}$ , then the classical Jackson inequality implies  $\Omega_r\left(f, \frac{1}{n}\right)_{pq, \omega} \geq c2^{-n}$  but inequality (8) yields  $\Omega_r\left(f, \frac{1}{n}\right)_{pq, \omega} \geq c2^{-n/2}$ .

An analogue of Theorem 1.5 for the space  $L^\infty$  was proved in [18]. In [2], the weak converse of (5)

$$\Omega_r\left(f, \frac{1}{n}\right)_{L_\omega^{pq}} \leq \frac{c}{n^{2r}} \sum_{\nu=0}^n (\nu + 1)^{2r-1} E_\nu(f)_{L_\omega^{pq}}, \tag{9}$$

for  $r \in \mathbb{N}$ ,  $f \in L_\omega^{pq}(\mathbb{T})$ ,  $\omega \in A_p$  and  $1 < p, q < \infty$  was obtained.

**Theorem 1.7.** Let  $1 < p < \infty$  and  $1 < q \leq 2$  or  $p > 2$  and  $q \geq 2$ ,  $\omega \in A_p$ ,  $f \in L_\omega^{pq}(\mathbb{T})$ . If  $n \in \mathbb{N}$ ,  $r \in \mathbb{R}^+$  and  $\gamma := \min\{2, q\}$ , then there is a constant  $c > 0$  only depending on  $r, q, p$  and  $C_{A_p}$  such that

$$\Omega_r\left(f, \frac{1}{n}\right)_{pq, \omega} \leq \frac{c}{n^{2r}} \left( \sum_{\nu=1}^n \nu^{2\gamma r-1} E_{\nu-1}^\gamma(f)_{pq, \omega} \right)^{1/\gamma}. \tag{10}$$

The analogues of this improved converse theorem were proven in [15] for  $r \in \mathbb{N}$ ,  $f \in L_\omega^p(\mathbb{T})$ ,  $1 < p < \infty$ ,  $\omega \in A_p$  with the modulus of smoothness  $W_r\left(f, \frac{1}{n}\right)_{L_\omega^p}$ ; in [1] for  $r \in \mathbb{R}^+$ ,  $f \in L_\omega^p(\mathbb{T})$ ,  $1 < p < \infty$ ,  $\omega \in A_p$  with the fractional modulus of smoothness (4); in [14] for  $r \in \mathbb{N}$ ,  $f \in L_\omega^{pq}(\mathbb{T})$ ,  $1 < p < \infty$  and  $1 < q \leq 2$  or  $p > 2$  and  $q \geq 2$ ,  $\omega \in A_p$  with  $W_r\left(f, \frac{1}{n}\right)_{L_\omega^{pq}}$ ; in [20] for  $r \in \mathbb{R}^+$ ,  $f \in L_\omega^{pq}(\mathbb{T})$ ,  $1 < p < \infty$

and  $1 < q \leq 2$  or  $p > 2$  and  $q \geq 2$ ,  $\omega \in A_p$  with a modulus of smoothness defined by Ky [16].

The inequality (10) is better than (9). Indeed, using the fact that  $x^\gamma$  is convex for  $\gamma = \min \{2, p\}$  we obtain that

$$\begin{aligned} & \left( \nu \nu^{2r-1} E_\nu(f)_{pq,\omega} \right)^\gamma - \left( (\nu - 1) \nu^{2r-1} E_\nu(f)_{pq,\omega} \right)^\gamma \\ & \leq \left( \sum_{\mu=1}^{\nu} \mu^{2r-1} E_\mu(f)_{pq,\omega} \right)^\gamma - \left( \sum_{\mu=1}^{\nu-1} \mu^{2r-1} E_\mu(f)_{pq,\omega} \right)^\gamma. \end{aligned}$$

Taking the summation over  $\nu$ , we obtain that

$$\begin{aligned} & \sum_{\nu=1}^n \left\{ \left( \nu \nu^{2r-1} E_\nu(f)_{pq,\omega} \right)^\gamma - \left( (\nu - 1) \nu^{2r-1} E_\nu(f)_{pq,\omega} \right)^\gamma \right\} \\ & \leq \sum_{\nu=1}^n \left\{ \left( \sum_{\mu=1}^{\nu} \mu^{2r-1} E_\mu(f)_{pq,\omega} \right)^\gamma - \left( \sum_{\mu=1}^{\nu-1} \mu^{2r-1} E_\mu(f)_{pq,\omega} \right)^\gamma \right\}, \end{aligned}$$

hence we have the inequality

$$\left\{ \sum_{\nu=1}^n \nu^{2\gamma r-1} E_{\nu-1}^\gamma(f)_{pq,\omega} \right\}^{1/\gamma} \leq 2 \sum_{\nu=1}^n \nu^{2r-1} E_{\nu-1}(f)_{pq,\omega}.$$

We give the Marcinkiewicz multiplier and Littlewood-Paley theorems in  $L_\omega^{pq}(\mathbb{T})$  which are used in the proofs of previous Theorems.

**Theorem 1.8.** *Let  $\lambda_0, \lambda_1, \dots$  be a sequence of real numbers such that*

$$|\lambda_l| \leq M \text{ and } \sum_{\nu=2^{l-1}}^{2^l-1} |\lambda_\nu - \lambda_{\nu+1}| \leq M,$$

for all  $\nu, l \in \mathbb{N}$ . If  $1 < p, q < \infty$ ,  $\omega \in A_p$  and  $f \in L_\omega^{pq}(\mathbb{T})$  with Fourier series  $\sum_{\nu=0}^\infty (a_\nu(f) \cos \nu x + b_\nu(f) \sin \nu x)$ , then there exists  $h \in L_\omega^{pq}(\mathbb{T})$  such that the series  $\sum_{\nu=0}^\infty \lambda_\nu (a_\nu(f) \cos \nu x + b_\nu(f) \sin \nu x)$  is the Fourier series of  $h$  and

$$\|h\|_{pq,\omega} \leq C \|f\|_{pq,\omega}, \tag{11}$$

where  $C$  does not depend on  $f$ .

**Theorem 1.9.** *Let  $\nu \in \mathbb{N}$ ,  $1 < p, q < \infty$ ,  $\omega \in A_p$  and  $f \in L_\omega^{pq}(\mathbb{T})$  with Fourier series  $\sum_{\nu=0}^\infty (a_\nu(f) \cos \nu x + b_\nu(f) \sin \nu x)$ , then there exist constants  $c_1$  and  $c_2$  independent of  $f$  such that*

$$c_1 \left\| \left( \sum_{\mu=\nu}^\infty |\Delta_\mu|^2 \right)^{1/2} \right\|_{pq,\omega} \leq \|f\|_{pq,\omega} \leq c_2 \left\| \left( \sum_{\mu=\nu}^\infty |\Delta_\mu|^2 \right)^{1/2} \right\|_{pq,\omega'}, \tag{12}$$

where

$$\Delta_\mu := \Delta_\mu(x, f) := \sum_{\nu=2^{\mu-1}}^{2^\mu-1} (a_\nu(f) \cos \nu x + b_\nu(f) \sin \nu x).$$

## 2 Proof of the main results

From [5, 14] we recall four important properties of the spaces  $L_\omega^{pq}(\mathbb{T})$ .

**Lemma A** ([5] or [13, prop. 5.1.2]) *For  $1 < p, q < \infty$ , there exists a  $c > 0$  such that for every  $f \in L_\omega^{pq}(\mathbb{T})$*

$$c^{-1} \|f\|_{pq,\omega} \leq \sup \left| \int_{\mathbb{T}} f(x)g(x)w(x)dx \right| \leq c \|f\|_{pq,\omega},$$

where the supremum is taken over all functions  $g$  for which  $\|g\|_{p'q',\omega} \leq 1$ .

**Lemma B [14].** *Let  $1 < p < \infty$  and  $1 < q \leq 2$ . Then for an arbitrary system of functions  $\{\varphi_j(x)\}_{j=1}^m, \varphi_j \in L_\omega^{pq}(\mathbb{T})$  we have*

$$\left\| \left( \sum_{j=1}^m \varphi_j^2 \right)^{1/2} \right\|_{pq,\omega} \leq c \left( \sum_{j=1}^m \|\varphi_j\|_{pq,\omega}^q \right)^{1/q}$$

with a constant  $c$  independent of  $\varphi_j$  and  $m$ .

**Lemma C [14].** *Let  $2 < p < \infty$  and  $q \geq 2$ . For an arbitrary system  $\{\varphi_j(x)\}_{j=1}^m, \varphi_j \in L_\omega^{pq}(\mathbb{T})$ , we have*

$$\left\| \left( \sum_{j=1}^m \varphi_j^2 \right)^{1/2} \right\|_{pq,\omega} \leq c \left( \sum_{j=1}^m \|\varphi_j\|_{pq,\omega}^2 \right)^{1/2}$$

with a constant  $c$  independent of  $\varphi_j$  and  $m$ .

**Lemma D [14].** *Let  $1 < p, q < \infty, f \in L_\omega^{pq}(\mathbb{T})$  and  $w \in A_p$ . If  $B_{k,\mu}(x) = a_k(f) \cos(k + \mu\frac{\pi}{2})x + b_k(f) \sin(k + \mu\frac{\pi}{2})x$ , where  $a_k, b_k$  are the Fourier coefficients of  $f$ , then*

$$\left\| \sum_{k=2^{i+1}}^{2^{i+1}} k^\mu B_{k,\mu} \right\|_{pq,\omega} \leq c 2^{i\mu} E_{2^i}(f)_{pq,\omega},$$

where the constant  $c$  is independent of  $f$  and  $i$ .

*Proof of Theorem 1.8.* We define a linear operator

$$Tf(x) := \sum_{\nu=0}^{\infty} \lambda_\nu (a_\nu(f) \cos \nu x + b_\nu(f) \sin \nu x)$$

for  $f \in L_\omega^{pq}(\mathbb{T})$  which is bounded (in particular is of weak type  $(p, p)$ ) in  $L_\omega^p(\mathbb{T})$  for every  $p > 1$  by [4, Th. 4.4]. Therefore the hypothesis of the interpolation theorem for Lorentz spaces [3, Th. 4.13] is fulfilled. Applying this theorem we get the desired result (11). ■

*Proof of Theorem 1.9.* Let us define a quasilinear operator

$$Tf(x) := \left( \sum_{\mu=\nu}^{\infty} |\Delta_\mu(x, f)|^2 \right)^{1/2}.$$



This operator is bounded in  $L^p_\omega(\mathbb{T})$  for every  $p > 1$  by [4, Th. 4.5]. Therefore the left hand side of the required result (12) is derived by means of the interpolation theorem for Lorentz spaces [3, Th. 4.13].

Using Hölder’s inequality for  $f \in L^{pq}_\omega(\mathbb{T}) \cap L^2_\omega(\mathbb{T})$ ,  $g \in L^{p'q'}_\omega(\mathbb{T}) \cap L^2_\omega(\mathbb{T})$  and the left hand side of (12) we obtain

$$\begin{aligned} \int_{\mathbb{T}} |f(x)g(x)| \omega(x)dx &= \int_{\mathbb{T}} \left| \sum_{\mu=1}^{\infty} \Delta_\mu(x, f)\Delta_\mu(x, g) \right| \omega(x)dx \\ &\leq \int_{\mathbb{T}} \sum_{\mu=1}^{\infty} |\Delta_\mu(x, f)\Delta_\mu(x, g)|\omega(x)dx \\ &\leq \int_{\mathbb{T}} \left[ \sum_{\mu=1}^{\infty} |\Delta_\mu(x, f)|^2 \right]^{1/2} \left[ \sum_{\mu=1}^{\infty} |\Delta_\mu(x, g)|^2 \right]^{1/2} \omega(x)dx \\ &\leq \left\| \left[ \sum_{\mu=1}^{\infty} |\Delta_\mu(x, f)|^2 \right]^{1/2} \right\|_{pq, \omega} \left\| \left[ \sum_{\mu=1}^{\infty} |\Delta_\mu(x, g)|^2 \right]^{1/2} \right\|_{p'q', \omega} \\ &\leq C \left\| \left[ \sum_{\mu=1}^{\infty} |\Delta_\mu(x, f)|^2 \right]^{1/2} \right\|_{pq, \omega} \|g\|_{p'q', \omega}. \end{aligned}$$

where  $p' = p / (p - 1)$ ,  $q' = q / (q - 1)$ . Taking the supremum in the last inequality over all functions  $g \in L^{p'q'}_\omega(\mathbb{T})$  satisfying  $\|g\|_{p'q', \omega} \leq 1$ , we find, applying Lemma A that

$$\|f\|_{pq, \omega} \leq C \left\| \left( \sum_{\mu=1}^{\infty} |\Delta_\mu|^2 \right)^{1/2} \right\|_{pq, \omega}.$$

The density of  $L^{pq}_\omega(\mathbb{T}) \cap L^2_\omega(\mathbb{T})$  in  $L^{pq}_\omega(\mathbb{T})$  yields the last inequality for any  $f \in L^{pq}_\omega(\mathbb{T})$ . ■

**Lemma 2.1.** *If  $0 < \alpha \leq \beta$ ,  $\omega \in A_p$ ,  $1 < p, q < \infty$  and  $f \in L^{pq}_\omega(\mathbb{T})$  then*

$$\Omega_\beta(f, \cdot)_{pq, \omega} \leq c \Omega_\alpha(f, \cdot)_{pq, \omega}. \tag{13}$$

*Proof.* The proof of Lemma 2.1 is similar to the proof of [1, Lemma 1]. ■

**Lemma 2.2.** *Let  $r \in \mathbb{R}^+$ ,  $1 < p, q < \infty$ ,  $\omega \in A_p$  and  $T_n \in \mathcal{T}_n$  for  $n = 1, 2, \dots$ . Then*

$$\Omega_r\left(T_n, \frac{1}{n}\right)_{pq, \omega} \preceq \frac{1}{n^{2r}} \left\| T_n^{(2r)} \right\|_{pq, \omega}$$

*holds with some constant only depending on  $r, p, q$  and  $C_{A_p}$ .*

*Proof.* For all  $x \geq 0$ , we have that

$$\left(1 - \frac{\sin x}{x}\right)_* \leq x^2,$$

where

$$\left(1 - \frac{\sin x}{x}\right)_* := \begin{cases} 1 - \frac{\sin x}{x} & \text{if } x \geq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

For  $0 < t$  and  $h_i \leq \frac{1}{n}$ , we have that

$$\begin{aligned} & \left\| \prod_{i=1}^{[r]} (I - \sigma_{h_i}) (I - \sigma_t)^{r-[r]} T_n \right\|_{pq,\omega} \\ &= \left\| \sum_{v=0}^n \left( 1 - \frac{\sin v h_1}{v h_1} \right)_* \cdots \left( 1 - \frac{\sin v h_{[r]}}{v h_{[r]}} \right)_* \left( 1 - \frac{\sin v t}{v t} \right)_*^{r-[r]} \right. \\ & \quad \left. A_v(T_n, x) \right\|_{pq,\omega} \\ &= \left\| \sum_{v=1}^n \frac{\left( 1 - \frac{\sin v h_1}{v h_1} \right)^2 (v h_1)^2 \cdots \left( 1 - \frac{\sin v h_{[r]}}{v h_{[r]}} \right)^2 (v h_{[r]})^2 \left( 1 - \frac{\sin v t}{v t} \right)^{r-[r]}}{(v h_1)^2 (v h_{[r]})^2 (v t)^2} \right. \\ & \quad \left. A_v(T_n, x) \right\|_{pq,\omega} \\ &\leq n^{-2r} \left\| \sum_{v=1}^n \frac{\left( 1 - \frac{\sin v h_1}{v h_1} \right)}{(v h_1)^2} v^2 \cdots \frac{\left( 1 - \frac{\sin v h_{[r]}}{v h_{[r]}} \right)}{(v h_{[r]})^2} v^2 \left( 1 - \frac{\sin v t}{v t} \right)^{r-[r]} \right. \\ & \quad \left. v^{2(r-[r])} A_v(T_n, x) \right\|_{pq,\omega} \\ &\leq n^{-2r} \left\| \sum_{v=1}^n v^{2r} \frac{\left( 1 - \frac{\sin v h_1}{v h_1} \right)}{(v h_1)^2} \cdots \frac{\left( 1 - \frac{\sin v h_{[r]}}{v h_{[r]}} \right)}{(v h_{[r]})^2} \left( 1 - \frac{\sin v t}{v t} \right)^{r-[r]} A_v(T_n, x) \right\|_{pq,\omega}. \end{aligned}$$

Applying Theorem 1.8 we obtain that

$$\left\| \prod_{i=1}^{[r]} (I - \sigma_{h_i}) (I - \sigma_t)^{r-[r]} T_n \right\|_{pq,\omega} \leq n^{-2r} \left\| \sum_{v=1}^n v^{2r} A_v(T_n, x) \right\|_{pq,\omega}.$$

For  $v = 1, 2, 3, \dots$  we have

$$A_v(T_n, x) = A_v\left(T_n, x + \frac{r\pi}{v}\right) \cos r\pi + A_v\left(\tilde{T}_n, x + \frac{r\pi}{v}\right) \sin r\pi,$$

where  $\tilde{T}_n$  is the Fourier conjugate of  $T_n$ . Therefore

$$\begin{aligned} & \left\| \prod_{i=1}^{[r]} (I - \sigma_{h_i}) (I - \sigma_t)^{r-[r]} T_n \right\|_{pq,\omega} \\ &\preceq n^{-2r} \left\| \sum_{v=1}^n v^{2r} \left( A_v\left(T_n, x + \frac{r\pi}{v}\right) \cos r\pi + A_v\left(\tilde{T}_n, x + \frac{r\pi}{v}\right) \sin r\pi \right) \right\|_{pq,\omega} \\ &\preceq n^{-2r} \left( \left\| \sum_{v=1}^n v^{2r} A_v\left(T_n, x + \frac{r\pi}{v}\right) \right\|_{pq,\omega} + \left\| \sum_{v=1}^n v^{2r} A_v\left(\tilde{T}_n, x + \frac{r\pi}{v}\right) \right\|_{pq,\omega} \right). \end{aligned}$$

Since

$$A_v(T_n^{(2r)}, x) = v^{2r} A_v\left(T_n, x + \frac{r\pi}{v}\right),$$

for  $\nu = 1, 2, 3, \dots$ , we find

$$\begin{aligned} & \Omega_r\left(T_n, \frac{1}{n}\right)_{pq,\omega} \\ & \leq n^{-2r} \left( \left\| \sum_{\nu=1}^n \nu^{2r} A_\nu\left(T_n, x + \frac{r\pi}{\nu}\right) \right\|_{pq,\omega} + \left\| \sum_{\nu=1}^n \nu^{2r} A_\nu\left(\tilde{T}_n, x + \frac{r\pi}{\nu}\right) \right\|_{pq,\omega} \right) \\ & \leq n^{-2r} \left( \left\| T_n^{(2r)} \right\|_{pq,\omega} + \left\| \tilde{T}_n^{(2r)} \right\|_{pq,\omega} \right) \leq n^{-2r} \left\| T_n^{(2r)} \right\|_{pq,\omega}. \quad \blacksquare \end{aligned}$$

**Lemma 2.3.** Let  $r \in \mathbb{R}_+$ ,  $1 < p, q < \infty$ ,  $\omega \in A_p$  and  $T_n \in \mathcal{T}_n$ . For  $n = 1, 2, \dots$ , we have that

$$\frac{1}{n^{2r}} \left\| T_n^{(2r)} \right\|_{pq,\omega} \leq \Omega_r\left(T_n, \frac{1}{n}\right)_{pq,\omega}$$

with some constant depending only on  $r, p, q$  and  $C_{A_p}$ .

*Proof.*

$$\begin{aligned} n^{-2r} \left\| T_n^{(2r)} \right\|_{pq,\omega} &= n^{-2r} \left\| \sum_{\nu=1}^n \nu^{2r} A_\nu\left(T_n, x + \frac{r\pi}{\nu}\right) \right\|_{pq,\omega} \\ &= n^{-2r} \left\| \sum_{\nu=1}^n \nu^{2r} \left( A_\nu(T_n, x) \cos r\pi + A_\nu(\tilde{T}_n, x) \sin r\pi \right) \right\|_{pq,\omega} \\ &\leq n^{-2r} \left\| \sum_{\nu=1}^n \nu^{2r} A_\nu(T_n, x) \cos r\pi \right\|_{pq,\omega} \\ &\quad + n^{-2r} \left\| \sum_{\nu=1}^n \nu^{2r} A_\nu(\tilde{T}_n, x) \sin r\pi \right\|_{pq,\omega} \\ &= \left\| \sum_{\nu=1}^n \cos r\pi \left( \frac{\left(\frac{\nu}{n}\right)^2}{1 - \frac{\sin \frac{\nu}{n}}{\frac{\nu}{n}}} \right)^r \left( 1 - \frac{\sin \frac{\nu}{n}}{\frac{\nu}{n}} \right)^r A_\nu(T_n, x) \right\|_{pq,\omega} \\ &\quad + \left\| \sum_{\nu=1}^n \sin r\pi \left( \frac{\left(\frac{\nu}{n}\right)^2}{1 - \frac{\sin \frac{\nu}{n}}{\frac{\nu}{n}}} \right)^r \left( 1 - \frac{\sin \frac{\nu}{n}}{\frac{\nu}{n}} \right)^r A_\nu(\tilde{T}_n, x) \right\|_{pq,\omega}. \end{aligned}$$

Applying Theorem 1.8 and the linearity of the conjugate operator we get

$$\begin{aligned} n^{-2r} \left\| T_n^{(2r)} \right\|_{pq,\omega} &\leq \left\| \sum_{\nu=1}^n \left( 1 - \frac{\sin \frac{\nu}{n}}{\frac{\nu}{n}} \right)^r A_\nu(T_n, x) \right\|_{pq,\omega} \\ &\quad + \left\| \sum_{\nu=1}^n \left( 1 - \frac{\sin \frac{\nu}{n}}{\frac{\nu}{n}} \right)^r A_\nu(\tilde{T}_n, x) \right\|_{pq,\omega} \\ &= \left\| \sum_{\nu=1}^n \left( 1 - \frac{\sin \frac{\nu}{n}}{\frac{\nu}{n}} \right)^r A_\nu(T_n, x) \right\|_{pq,\omega} \\ &\quad + \left\| \left( \sum_{\nu=1}^n \left( 1 - \frac{\sin \frac{\nu}{n}}{\frac{\nu}{n}} \right)^r A_\nu(T_n, x) \right)^\sim \right\|_{pq,\omega}. \end{aligned}$$

From the boundedness of the conjugate operator [14] we have

$$\begin{aligned} & n^{-2r} \left\| T_n^{(2r)} \right\|_{pq,\omega} \\ & \preceq \left\| \sum_{\nu=1}^n \left( 1 - \frac{\sin \frac{\nu}{n}}{\frac{\nu}{n}} \right)^r A_\nu(T_n, x) \right\|_{pq,\omega} + \left\| \sum_{\nu=1}^n \left( 1 - \frac{\sin \frac{\nu}{n}}{\frac{\nu}{n}} \right)^r A_\nu(T_n, x) \right\|_{pq,\omega} \\ & \preceq \left\| \left( I - \sigma_{\frac{1}{n}} \right)^r T_n \right\|_{pq,\omega} \preceq \sup_{0 < h_i, u < 1/n} \left\| \prod_{i=1}^{[r]} (I - \sigma_{h_i}) (I - \sigma_u)^{r-[r]} T_n \right\|_{pq,\omega} \\ & \preceq \Omega_r \left( T_n, \frac{1}{n} \right)_{pq,\omega}. \end{aligned} \quad \blacksquare$$

*Proof of Theorem 1.2.* From Lemma 2.1 and [2, Th. 1.1] we have

$$E_n(f)_{pq,\omega} \leq c \Omega_{[r]+1} \left( f, \frac{1}{n+1} \right)_{pq,\omega} \leq C \Omega_r \left( f, \frac{1}{n+1} \right)_{pq,\omega}$$

for  $n + 1 \in \mathbb{N}$  and the assertion (5) follows. ■

**Lemma 2.4.** Let  $1 < p, q < \infty$ ,  $\omega \in A_p$ ,  $f \in L_\omega^{pq}(\mathbb{T})$  and  $\gamma > 0$ . Then for any  $0 < t < 2$ ,

$$\Omega_\gamma(f, t)_{pq,\omega} \preceq t^\gamma \left\| f^{(\gamma)} \right\|_{pq,\omega}.$$

*Proof.* There is some  $n = 1, 2, 3, \dots$  such that  $(1/n) < t \leq (2/n)$ . From Lemma 2.2 we get

$$\Omega_\gamma(f, t)_{pq,\omega} \leq \Omega_\gamma(f - T_n, t)_{pq,\omega} + \Omega_\gamma(T_n, t)_{pq,\omega} \preceq E_n(f)_{pq,\omega} + t^{2\gamma} \left\| T_n^{(2\gamma)} \right\|_{pq,\omega}.$$

On the other hand applying [20, (3.9) and Th. 1.3] and Theorem 1.2 we have

$$E_n(f)_{pq,\omega} \preceq \frac{1}{n^{2\gamma}} E_n(f^{(2\gamma)})_{pq,\omega} \preceq \frac{1}{n^{2\gamma}} \Omega_\gamma \left( f^{(2\gamma)}, \frac{1}{n} \right)_{pq,\omega} \preceq t^{2\gamma} \left\| f^{(2\gamma)} \right\|_{pq,\omega}.$$

Using Theorem 1.2 and [20, Th. 1.3] the proof is completed. ■

*Proof of Theorem 1.3.* We have to show that (6) holds. Let  $T_n$  be the near best approximating trigonometric polynomial to  $f$ . From Theorem 1.2

$$\|f - T_n\|_{pq,\omega} \preceq E_n(f)_{pq,\omega} \leq c \Omega_r \left( f, \frac{1}{n+1} \right)_{pq,\omega}.$$

Applying Lemma 2.3, we find that

$$\begin{aligned} & \frac{1}{n^{2r}} \left\| T_n^{(2r)} \right\|_{pq,\omega} \preceq \Omega_r \left( T_n, \frac{1}{n} \right)_{pq,\omega} \preceq \Omega_r \left( T_n - f, \frac{1}{n} \right)_{pq,\omega} + \Omega_r \left( f, \frac{1}{n} \right)_{pq,\omega} \\ & \preceq \left\| f - T_n \right\|_{pq,\omega} + \Omega_r \left( f, \frac{1}{n} \right)_{pq,\omega} \preceq \Omega_r \left( f, \frac{1}{n} \right)_{pq,\omega} \end{aligned}$$

and

$$\|f - T_n\|_{pq,\omega} + \frac{1}{n^{2r}} \|T_n^{(2r)}\|_{pq,\omega} \leq \Omega_r\left(f, \frac{1}{n}\right)_{pq,\omega}.$$

On the other hand using Lemma 2.2

$$\begin{aligned} \Omega_r\left(f, \frac{1}{n}\right)_{pq,\omega} &\leq \Omega_r\left(f - T_n, \frac{1}{n}\right)_{pq,\omega} + \Omega_r\left(T_n, \frac{1}{n}\right)_{pq,\omega} \\ &\leq \|f - T_n\|_{pq,\omega} + \frac{1}{n^{2r}} \|T_n^{(2r)}\|_{pq,\omega} = R_{2r}\left(f, \frac{1}{n}, L_\omega^{pq}\right). \end{aligned}$$

This completes the proof of (6). Using Lemma 2.4, properties of the modulus of smoothness and of the  $K$ -functional (7) are proven.

*Proof of Theorem 1.5.* By Corollary 1.4 we have for  $v \leq n$

$$\Omega_r(f, 1/v)_{pq,\omega} \leq (1 + n/v)^{2r} \Omega_r(f, 1/n)_{pq,\omega}$$

and

$$\prod_{v=1}^n \Omega_r(f, 1/v)_{pq,\omega} \leq \prod_{v=1}^n (1 + n/v)^{2r} \left(\Omega_r(f, 1/n)_{pq,\omega}\right)^n.$$

For every  $n$  we have

$$\prod_{v=1}^n (1 + n/v)^{2r} \leq \left(\frac{2n}{\sqrt[n]{n!}}\right)^{2r}.$$

Using Stirling's formula

$$n! \asymp \sqrt{2\pi n} n^n e^{-n} e^{\theta(n)} \text{ with } |\theta(n)| \leq 1/(12n)$$

we get

$$\prod_{v=1}^n (1 + n/v)^{2r} \leq 2^{2r} e^{4r}.$$

Thus

$$\left(\prod_{v=1}^n \Omega_r(f, 1/v)_{pq,\omega}\right)^{1/n} \leq c \Omega_r(f, 1/n)_{pq,\omega}.$$

From (5) and the property  $E_n(f)_{pq,\omega} \rightarrow 0$  as  $n \rightarrow \infty$  we find

$$\left(\prod_{v=1}^n E_v(f)_{pq,\omega}\right)^{1/n} \leq \left(\prod_{v=1}^n \Omega_r(f, 1/v)_{pq,\omega}\right)^{1/n} \leq c \Omega_r(f, 1/n)_{pq,\omega}.$$

*Proof of Theorem 1.7.* For  $1 < p, q < \infty$ ,  $\omega \in A_p$ , let  $f \in L_\omega^{pq}(\mathbb{T})$  be such that  $\int_0^{2\pi} f(x) dx = 0$ . We assume that  $f$  has Fourier series (1). We choose  $m \in \mathbb{N}$  such that  $2^m \leq n < 2^{m+1}$ . Let us denote  $S_n(x) := S_n(x, f) := \sum_{k=0}^n A_k(f, x)$ , for  $x \in \mathbb{T}$ , where  $A_k(f, x) = a_k(f) \cos kx + b_k(f) \sin kx$ . By [14, Prop. 3.4], we have that

$$\|f - S_n\|_{pq,\omega} \leq c E_n(f)_{pq,\omega}. \quad (14)$$

It is well-known that  $\sigma_{t,h_1,h_2,\dots,h_{[r]}}^r f := \prod_{i=1}^{[r]} (I - \sigma_{h_i}) (I - \sigma_t)^{r-[r]} f$  has Fourier series

$$\begin{aligned} & \sigma_{t,h_1,h_2,\dots,h_{[r]}}^r f(\cdot) \\ & \sim \sum_{\nu=0}^{\infty} \left(1 - \frac{\sin \nu t}{\nu t}\right)_*^{r-[r]} \left(1 - \frac{\sin \nu h_1}{\nu h_1}\right)_* \dots \left(1 - \frac{\sin \nu h_{[r]}}{\nu h_{[r]}}\right)_* A_{\nu}(f, x). \end{aligned}$$

Moreover

$$\begin{aligned} & \sigma_{t,h_1,h_2,\dots,h_{[r]}}^r f(\cdot) \\ & = \sigma_{t,h_1,h_2,\dots,h_{[r]}}^r (f(\cdot) - S_{2^{m-1}}(\cdot, f)) + \sigma_{t,h_1,h_2,\dots,h_{[r]}}^r S_{2^{m-1}}(\cdot, f). \end{aligned}$$

From (14) and  $E_n(f)_{p,\omega} \rightarrow 0$  we have

$$\begin{aligned} & \left\| \sigma_{t,h_1,h_2,\dots,h_{[r]}}^r (f(\cdot) - S_{2^{m-1}}(\cdot, f)) \right\|_{pq,\omega} \\ & \leq c \|f(\cdot) - S_{2^{m-1}}(\cdot, f)\|_{pq,\omega} \leq c E_{2^{m-1}}(f)_{pq,\omega} \\ & \leq \frac{c}{n^{2r}} \left\{ \sum_{\nu=1}^n \nu^{2\gamma r-1} E_{\nu-1}^{\gamma}(f)_{p,\omega} \right\}^{1/\gamma}. \end{aligned}$$

On the other hand, it follows from (12) that

$$\left\| \sigma_{t,h_1,h_2,\dots,h_{[r]}}^r S_{2^{m-1}}(\cdot, f) \right\|_{pq,\omega} \leq c \left\| \left\{ \sum_{\mu=1}^m |\delta_{\mu}|^2 \right\}^{1/2} \right\|_{pq,\omega}$$

where

$$\delta_{\mu} := \sum_{\nu=2^{\mu-1}}^{2^{\mu}-1} \left(1 - \frac{\sin \nu t}{\nu t}\right)^{r-[r]} \left(1 - \frac{\sin \nu h_1}{\nu h_1}\right) \dots \left(1 - \frac{\sin \nu h_{[r]}}{\nu h_{[r]}}\right) A_{\nu}(f, x).$$

By Lemmas B and C

$$\left\| \left\{ \sum_{\mu=1}^m |\delta_{\mu}|^2 \right\}^{1/2} \right\|_{pq,\omega} \leq \left\{ \sum_{\mu=1}^m \|\delta_{\mu}\|_{pq,\omega}^{\gamma} \right\}^{1/\gamma}.$$

By Abel's transformation we obtain

$$\begin{aligned} & \|\delta_{\mu}\|_{pq,\omega} \\ & \leq \sum_{\nu=2^{\mu-1}}^{2^{\mu}-2} \left| \left(1 - \frac{\sin \nu t}{\nu t}\right)^{r-[r]} \left(1 - \frac{\sin \nu h_1}{\nu h_1}\right) \dots \left(1 - \frac{\sin \nu h_{[r]}}{\nu h_{[r]}}\right) \right. \\ & \quad \left. - \left(1 - \frac{\sin(\nu+1)t}{(\nu+1)t}\right)^{r-[r]} \left(1 - \frac{\sin(\nu+1)h_1}{(\nu+1)h_1}\right) \right. \\ & \quad \left. \dots \left(1 - \frac{\sin(\nu+1)h_{[r]}}{(\nu+1)h_{[r]}}\right) \right| \left\| \sum_{l=2^{\mu-1}}^{\nu} A_l(f, x) \right\|_{pq,\omega} \end{aligned}$$

$$+ \left| \left( 1 - \frac{\sin(2^\mu - 1)t}{(2^\mu - 1)t} \right)^{r-[r]} \left( 1 - \frac{\sin(2^\mu - 1)h_1}{(2^\mu - 1)h_1} \right) \dots \left( 1 - \frac{\sin(2^\mu - 1)h_{[r]}}{(2^\mu - 1)h_{[r]}} \right) \right| \left\| \sum_{l=2^{\mu-1}}^{2^\mu-1} A_l(f, x) \right\|_{pq, \omega}$$

and by Lemma D

$$\left\| \sum_{l=2^{\mu-1}}^v A_l(f, x) \right\|_{pq, \omega} \leq c E_{2^{\mu-1}-1}(f)_{pq, \omega}$$

and

$$\left\| \sum_{l=2^{\mu-1}}^{2^\mu-1} A_l(f, x) \right\|_{pq, \omega} \leq C E_{2^{\mu-1}-1}(f)_{pq, \omega}.$$

Since  $x^r \left( 1 - \frac{\sin x}{x} \right)^r$  is non decreasing for positive  $x$  we have

$$\|\delta_\mu\|_{pq, \omega} \leq c 2^{2\mu r} t^{2(r-[r])} h_1^2 h_2^2 \dots h_{[r]}^2 E_{2^{\mu-1}-1}(f)_{pq, \omega}$$

and hence

$$\begin{aligned} & \left\| \sigma_{t, h_1, h_2, \dots, h_{[r]}}^r S_{2^{m-1}}(\cdot, f) \right\|_{pq, \omega} \\ & \leq ct^{2(r-[r])} h_1^2 h_2^2 \dots h_{[r]}^2 \left\{ \sum_{\mu=1}^m 2^{\mu r \gamma} E_{2^{\mu-1}-1}(f)_{pq, \omega} \right\}^{1/\gamma} \\ & \leq ct^{2(r-[r])} h_1^2 h_2^2 \dots h_{[r]}^2 \left\{ 2^{\gamma r} E_0^\gamma(f)_M \right\}^{1/\alpha} \\ & + ct^{2(r-[r])} h_1^2 h_2^2 \dots h_{[r]}^2 \left\{ \sum_{\mu=2}^m \sum_{\nu=2^{\mu-2}}^{2^{\mu-1}-1} \nu^{2\gamma r-1} E_{\nu-1}^\gamma(f)_{pq, \omega} \right\}^{1/\gamma} \\ & \leq ct^{2(r-[r])} h_1^2 h_2^2 \dots h_{[r]}^2 \left\{ \sum_{\nu=1}^{2^{m-1}-1} \nu^{2\gamma r-1} E_{\nu-1}^\gamma(f)_{pq, \omega} \right\}^{1/\gamma}. \end{aligned}$$

Therefore we find

$$\Omega_r\left(f, \frac{1}{n}\right)_{pq, \omega} \leq \frac{c}{n^{2r}} \left\{ \sum_{\nu=1}^n \nu^{2\gamma r-1} E_{\nu-1}^\gamma(f)_{pq, \omega} \right\}^{1/\gamma}$$

finishing the proof of Theorem 1.7. ■

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