Grobner–Shirshov bases and embedding of a semigroup in a group

GRÖBNER-SHIRSHOV BASES AND EMBEDDING OF A SEMIGROUP IN A GROUP

EYLEM G. KARPUZ, FIRAT ATEŞ, A. SINAN ÇEVIK, AND JÖRG KOPPITZ

Abstract. The main goal of this paper is to show that if a group G has a Gröbner-Shirshov basis $\Re$ that satisfies the condition R_{+}, then the semigroup P (with positive rules in $\Re$ as a defining relation) embeds in this group G. As a consequence of our result, we obtain that the semigroup B_{n+1}^{+} of braids can be embedded in the braid group.

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1. Introduction and Preliminaries

It is well known that a semigroup P *embeds* in a group G if there exists a monomorphism from P into G, and then a semigroup P is embeddable into a group, or is *group-embeddable*, if there exists some group G into which P embeds. There have been many research papers for the investigation of necessary or/and sufficient conditions of a semigroup to be embeddable into a group. Cancellation is certainly a necessary condition for a semigroup to be embeddable into a group. Therefore it was asked whether all cancellative semigroups are group-embeddable. Malcev [28] answered this question in a negative manner, and later established a necessary and sufficient condition for a semigroup to be group-embeddable ([29]). A further necessary and sufficient condition for group-embeddable result was given by Lambek ([30]). In fact the result basically states that *a semigroup can be embedded in a group if and only if the cancellation law and the polyhedral condition are satisfied.* Additionally, there exists another embedding result which originally belongs to Ore ([32]). Although Ore's result was firstly stated as a sufficient condition for embedding of rings without zero divisors into division rings, it can be easily adapted to embedding of semigroups into groups. The Ore's theorem ([32]) says that *any left- or right-reversible cancellative semigroup embeds in a group.*

Among these above theorems, there also exists a quite important result presented by Adjan ([1]) that gives a sufficient condition for the group-embeddability. According to Adjan, *for a semigroup P having a presentation $Sgp\langle A\ ;\ R\rangle$, if left and right graphs of the presentation $Sgp\langle A\ ;\ R\rangle$ contain no non-trivial cycles, then the semigroup P is embeddable in a group* (see [1] for the details). A geometric proof of Adjan's Theorem is due to Remmers

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[33]. After that Stallings [35] used a graph-theoretical lemma to give another proof of this important result. Finally Kashintsev [26] and Guba [25] studied small cancellation theory to generalize Adjan's Theorem.

In this paper, as a main result, we establish the embeddability of a semigroup in a group via the Gröbner-Shirshov basis theory (see Theorem 2.5 below). After that, by considering braid groups, we present two examples that satisfy our main result. In the literaure there are some remarkable works on "embeddability by using Gröbner-Shirshov bases". A most difficult proof using these ideas was given by Bokut in four long papers [11, 12, 13, 14]. In a joint work ([15]), Bokut et al. also proved that in the following cases, each (resp. countably generated) algebra can be embedded into a simple (resp. two generated) algebra: associative differential algebras, associative Ω-algebras, associative λ-algebras. Additionally above works, again in a joint paper [18], Bokut and Shum defined the notion of a "*relative Gröbner-Shirshov basis*" which can be also related to embedding of semigroups into groups. We may finally refer [7, 8] to the reader for some other works on embeddability.

Let us continue by introducing the fundamental facts on the *Gröbner-Shirshov basis*. Let K be a field and $K\langle X\rangle$ be the free associative algebra over K generated by X. Denote X^* the free monoid generated by X, where the empty word is the identity which is denoted by 1. For a word $w \in X^*$, let us denote the length of w by $|w|$. Let $(X^*, <)$ be a well ordered set. (It is clear that one can extend "$<$" to the algebra $K\langle X\rangle$ since the basis of it is X). Then every nonzero polynomial $f \in K\langle X\rangle$ has a leading word $\overline{f}$. If the coefficient of $\overline{f}$ in f is equal to 1_K, then f is called monic. Now suppose that f and g are two monic polynomials in $K\langle X\rangle$. Then there are two kinds of compositions:

(1) If w is a word such that $w = \overline{f}b = a\overline{g}$ for some $a, b \in X^*$ with $|\overline{f}| + |\overline{g}| > |w|$, then the polynomial $(f, g)_w = fb - ag$ is called the i*ntersection composition* of f and g with respect to w. The word w is called an *ambiguity* of intersection.
(2) If $w = \overline{f} = a\overline{g}b$ for some $a, b \in X^*$, then the polynomial $(f, g)_w = f - agb$ is called the *inclusion composition* of f and g with respect to w. The word w is called an *ambiguity* of inclusion.

Let $S \subseteq K\langle X\rangle$ with each $s \in S$ is monic. Then the composition $(f, g)_w$ is called *trivial modulo* (S, w) or simply just *trivial* if $(f, g)_w = \sum \alpha_i a_i s_i b_i$, where each $\alpha_i \in K$, $a_i, b_i \in X^*$, $s_i \in S$ and $a_i\overline{s_i}b_i < w$. If this is the case, then we write $(f, g)_w \equiv 0\ mod(S, w)$. In general, for $p, q \in K\langle X\rangle$, we write $p \equiv q\ mod(S, w)$ which means that $p - q = \sum \alpha_i a_i s_i b_i$, where each $\alpha_i \in K$, $a_i, b_i \in X^*$, $s_i \in S$ and $a_i\overline{s_i}b_i < w$.

By [9], the subset S (of the algebra $K\langle X\rangle$) endowed with the well ordering $<$ is called a Gröbner-Shirshov set (basis) if every composition $(f, g)_w$ of elements (polynomials) in S is trivial. This definition goes back to A. I. Shirshov (1962) [34]. Moreover, a well ordered $<$ on X^* is called *monomial* if we have

$$u < v \Rightarrow w_1uw_2 < w_1vw_2,$$

for all $u, v, w_1, w_2 \in X^*$.

The following key lemma was proved by Shirshov [34] for free Lie algebras with deg-lex ordering (see also [6]). Later, in [7], Bokut specialized the Shirshov's approach to associative algebras (see also [4]). Meanwhile, for commutative polynomials, this important lemma is known as the Buchberger's Theorem (see [20, 21]).

Now, for the field K, let $Id(S)$ be the ideal of $K\langle X\rangle$ generated by S. A word $u \in X^*$ is called *S-irreducible* if $u \neq a\bar{s}b$ for all $s \in S$ and all $a, b \in X^*$. Let $Irr(S)$ be the set of all S-irreducible words.

Lemma 1.1. (**Composition-Diamond Lemma**) *Let $S \subseteq K\langle X\rangle$ and let $<$ be a monomial order on X^*. Then the following statements are equivalent:*

1. *S is a Gröbner-Shirshov basis in $K\langle X\rangle$.*
2. *$f \in Id(S) \Rightarrow \overline{f} = a\bar{s}b$ for some $s \in S$ and $a, b \in X^*$.*
3. *$Irr(S)$ is a linear basis for the algebra $K\langle X|S\rangle := K\langle X\rangle/Id(S)$ generated by X with defining relations S.*

If a subset S of $K\langle X\rangle$ is not a Gröbner-Shirshov basis, then we can add all nontrivial compositions of polynomials of S into S. Then, by continuing this process (maybe infinitely) many times, we eventually obtain a Gröbner-Shirshov basis S^{comp}, and then such a process is called the *Shirshov algorithm*. We note that if S is a set of "semigroup relations" (that is, the polynomials of the form $u - v$, where $u, v \in X^*$), then any nontrivial composition will have the same form. As a result of this, the set S^{comp} also consists of semigroup relations.

Let $P = Sgp\,\langle X; S\rangle$ be a semigroup presentation. Then S is a subset of $K\langle X\rangle$ and hence one can obtain a Gröbner-Shirshov basis S^{comp}. In fact the last set does not depend on K and, as mentioned in the previous paragraph, it consists of semigroup relations. Thus S^{comp} will be called a *Gröbner-Shirshov basis* of P. This is the same as a Gröbner-Shirshov basis of the semigroup algebra $K\langle P\rangle = K\langle Sgp\langle X; S\rangle\rangle$ $(=: KP)$. We finally note that if S is a Gröbner-Shirshov basis of the semigroup $P = Sgp\,\langle X; S\rangle$, it follows from Lemma 1.1 that in this case any word $u \in X^*$ is equal in P to a unique S-irreducible word, called the *normal form* of u.

We finally note that if $G = gp\,\langle X; S\rangle$ is a group, then $G = Sgp\,\left\langle X \cup X^{-1}; S_0\right\rangle$, where

$$S_0 := S \cup \{xx^{-1},\ x^{-1}x = 1 \mid x \in X\}.$$

Then S_0 is called a Gröbner-Shirshov basis for the group G if it is a Gröbner-Shirshov basis for G as semigroup. We refer the papers [3, 16, 27] for studies over Gröbner-Shirshov bases of some semigroups.

After these all above introductory and preliminary material, in the next section, we will state and prove the main theorem of this paper.

2. The Main Result

Let G be a group presented by $G = gp < X; R >$ and let $\Re$ be a Gröbner-Shirshov basis for G. For a polynomial $u - v$ in $\Re$, if the monomials u and v are both positive then we call them *positive leading monomial* and *positive remainder*, respectively. (For example the polynomial $x_1x_2 - x_2x_1$ has both positive leading monomial x_1x_2 and positive remainder x_2x_1. However the polynomial $x_1x_2^{-1} - x_2x_1$ has not a positive leading monomial, it has actually just a positive remainder.) We say that $\Re$ *satisfies the condition* R_+ if each polynomial in $\Re$ which has a positive leading monomial has also a positive remainder. In addition, we denote by $\Re^+$ the subset of all polynomials in $\Re$ having positive leading monomial and positive remainder.

The following four lemmas will be needed for the proof of the main result (Theorem 2.5 below) of this paper. Before presenting them, let us assume that P is a semigroup generated by the same set X as G.

Lemma 2.1. *Let us suppose that the group G has a Gröbner-Shirshov basis $\Re$ in the free associative algebra $K\langle X\rangle$ with deg-lex ordering on X^*, and let $\Re$ satisfies the condition R_+. If $\Re^+$ is a set of defining relations for the semigroup P, then $\Re^+$ is a Gröbner-Shirshov basis for P.*

Assume that G has a Gröbner-Shirshov basis $\Re = \{x_1x_2 = x_2^{-1}x_1^{-1}, x_1x_3 = x_3x_1, x_2x_3 = x_3x_2, x_ix_i^{-1} = 1, x_i^{-1}x_i = 1\ (i = 1, 2, 3)\}$ such that the set $\Re^+ = \{x_1x_3 = x_3x_1, x_2x_3 = x_3x_2\}$ is a Gröbner-Shirshov basis for the semigroup P. In here, it is clear that $\Re$ does not have the condition R_+ and so this discussion shows that the converse of Lemma 2.1 is not true.

Proof. We need to prove that all compositions obtained by polynomials in $\Re^+$ are trivial. To do that let us take two polynomials $f = u_1 - v_1$ and $g = u_2 - v_2$ in $\Re^+$ and consider the intersection composition of f and g. Then we get the ambiguity $w = \widetilde{u_1}x'\underline{u_2}$ $(x' \in X)$, where $\widetilde{u_1}$ and $\underline{u_2}$ denote the words which do not have the last generator of u_1 and the first generator of u_2, respectively. Now we consider the inclusion composition of f and g. Hence we get the ambiguity $w' = u_1 = au_2b$ for some $a, b \in X^*$. Since $\Re$ is a Gröbner-Shirshov basis in the free associative algebra $K\langle X\rangle$ with deg-lex ordering on X^*, $(f, g)_w$ and $(f, g)_{w'}$ are trivial *modulo*$(\Re, w)$ and *modulo*$(\Re, w')$, respectively. In addition since $f = u_1 - v_1$ and $g = u_2 - v_2$ belong to $\Re^+$ the monomials u_1 and u_2 are positive monomials. So the polynomials used to reduce to zero the compositions $(f, g)_w$ and $(f, g)_{w'}$ belong to $\Re^+$. Therefore $\Re^+$ is a Gröbner-Shirshov basis for the semigroup P. □

Lemma 2.2. *Let us suppose that the group G has a Gröbner-Shirshov basis $\Re$ in the free associative algebra $K\langle X\rangle$ with deg-lex ordering on X^* and let $\Re$ satisfies the condition R_+. Assume that m_1 and m_2 are positive monomials. Then $m_1 - m_2 \in Id(\Re)$ if and only if $m_1 - m_2 \in Id(\Re^+)$.*

Proof. Let $m_1 - m_2 \in Id(\Re^+)$. Since $\Re^+ \subseteq \Re$, we get $m_1 - m_2 \in Id(\Re)$.

Conversely, let $m_1 - m_2 \in Id(\Re)$. Then there is an $m_1^* - m_2^* \in \Re$ and $a, b \in X^*$ such that $m_1 = am_1^*b$ by Lemma 1.1. Since m_1 is a positive monomial, and $\Re$ have the condition R_+, the monomial m_2^* is positive. Hence $m_1^* - m_2^* \in \Re^+$. This shows that $m_1 - m_2 \in Id(\Re^+)$, as required. □

Now, let us present the following other two lemmas which the proofs of them are quite clear.

Lemma 2.3. *Th esemigroup P embeds into G if and only if for all positive monomials m and n the following condition holds: "If m and n are equal in G, then m and n are equal in P".*

Lemma 2.4. *If $\Re$ is a Gröbner-Shirshov basis for the group $G = gp\langle X; R\rangle$ (for the semigroup $P = Sgp\langle X; R\rangle$), then $m_1 - m_2 \in Id(\Re)$ if and only if m_1 and m_2 are equal in G (in P).*

After all, the main result of this paper can be stated as in the following.

Theorem 2.5. *Suppose that the group $G = gp\langle X; R\rangle$ has a Gröbner-Shirshov basis $\Re$ in the free associative algebra $K\langle X\rangle$ with deg-lex ordering on X^*. Let $\Re$ satisfies the condition R_+. Then the semigroup $P = Sgp\langle X; \Re^+\rangle$ embeds into G.*

Proof. Let us suppose that $\Re^+$ is a basis for the semigroup P. Then, by Lemma 2.1, $\Re^+$ is a Gröbner-Shirshov basis for P in the free associative algebra $K\langle X\rangle$ with deg-lex ordering on X^*.

Now assume that m_1 and m_2 are two positive monomials such that they are equal in the group G. To complete the proof, by Lemma 2.3, we need to show that m_1 and m_2 are also equal in the semigroup P.

Since $\Re$ is a Gröbner-Shirshov basis for G, the equality of m_1 and m_2 in G implies that $m_1 - m_2 \in Id(\Re)$ (by Lemma 2.4). Since m_1 and m_2 are positive monomials, by Lemma 2.2, we get $m_1 - m_2 \in Id(\Re^+)$. Also since $\Re^+$ is a Gröbner-Shirshov basis for the semigroup P, the monomials m_1 and m_2 are actually equal to each other in P (by Lemma 2.4).

Hence the result. □

3. Applications over B_{n+1}

There is a range of papers determining the Gröbner-Shirshov basis of groups. If we point out in any case that the Gröbner-Shirshov basis satisfy the condition R_+, then we can establish a semigroup which can be embedded in the given group. In this section, we will do it for the *braid group* B_{n+1} (or B_n). Thus the aim of this section is to strengthen Theorem 2.5 by processing the group B_{n+1}.

In fact, the braid group B_{n+1} on $n + 1$ strands was intented by Artin in [2] and, again in the same reference, by defining a presentation

$$B_{n+1} = gp < a_1, \cdots, a_n \quad ; \quad a_{i+1}a_ia_{i+1} = a_ia_{i+1}a_i \ (1 \leq i < n), \\ a_ka_s = a_sa_k \ (k - s > 1) >,$$

it was shown that the word problem over B_{n+1} is solvable. After that, in [9], Bokut obtained a Gröbner-Shirshov basis for the braid group B_{n+1} in the *Artin-Garside generators* a_i $(1 \leq i \leq n)$, Δ, Δ^{-1}, where $\Delta = \delta_1\delta_2\cdots\delta_n$ with $\delta_i = a_i\cdots a_1$ ([24]). In fact, it has been assumed an ordering $\Delta^{-1} < \Delta < a_1 < \cdots < a_n$ among these generators. Moreover, again in [9], Bokut ordered words in this alphabet in the deg-lex way comparing two words first by their lengths and then lexicographically when the lengths are equal. (We should note that the empty word is the least element in the deg-lex order.) By $V(j, i)$, $W(j, i)$,$\cdots$, where $j \leq i$, it is understood that they are positive words having the letters $a_j, a_{j+1}, \cdots, a_i$. Also $V(i + 1, i) = 1$, $W(i+1, i) = 1, \cdots$. Given $V = V(1, i)$, let $V^{(k)}$ $(1 \leq k \leq n-i)$ be the result of shifting in V all indices of all letters by k, $a_1 \to a_{k+1}$, $\cdots$, $a_i \to a_{k+i}$. Also let us use the notation $V^{(1)} = V'$ and let us write $a_{ij} = a_ia_{i-1}\cdots a_j$ $(j \leq i - 1)$, $a_{ii} = a_i$ and $a_{ii+1} = 1$.

After all, the main result in [9] can be restated as in the following.

Theorem 3.1. ([9]) *A Gröbner-Shirshov basis of B_{n+1} in the Artin-Garside generators consists of the following relations:*

(1) $a_{i+1}a_iV(1,i-1)W(j,i)a_{i+1j} = a_ia_{i+1}a_iV(1,i-1)a_{ij}W(j,i)'$,
(2) $a_sa_k = a_ka_s, \quad s-k \geq 2$,
(3) $a_1V_1a_2a_1V_2\cdots V_{n-1}a_n\cdots a_1 = \Delta V_1^{(n-1)}V_2^{(n-2)}\cdots V'_{n-1}$,
(4) $a_l\Delta = \Delta a_{n-l+1}, \quad 1 \leq l \leq n$,
(5) $a_l\Delta^{-1} = \Delta^{-1}a_{n-l+1}, \quad 1 \leq l \leq n$,
(6) $\Delta\Delta^{-1} = 1, \ \Delta^{-1}\Delta = 1$,

where $1 < i \leq n-1$, $1 \leq j \leq i+1$, $V_i = V_i(1,i)$, and finally $W(j,i)$ and $W(j,i)'$ begin with a_i if it is not empty.

Hence the braid group B_{n+1} in the Artin-Garside generators has a Gröbner-Shirshov basis $\Re_{B_{n+1}}$ as given in the conditions (1)-(6) of Theorem 3.1. Further it is easy to see that $\Re_{B_{n+1}}$ satisfies the condition R_+. Now, as in [17], let us consider the braid semigroup B^+_{n+1} in the Artin generators a_i. In the same reference, it is proved that a Gröbner-Shirshov basis $\Re_{B^+_{n+1}}$ of B^+_{n+1} consists of only the relations (1) and (2) of Theorem 3.1. In other words, there does not exists any Δ generators in this basis since Δ is actually in the set of the Garside generators (we may refer [9] and [17] for all details). So, by our main result (Theorem 2.5), the braid semigroup B^+_{n+1} is embedded into the braid group B_{n+1}. Therefore Theorem 2.5 covers the following fact which gives a direct expression for this above embedding.

Corollary 3.2. ([24]) *The semigroup of positive braids B^+_{n+1} can be embedded into a group.*

Next, let us consider the braid group B_n on n-strands in the *Birman-Ko-Lee generators* enriched with the new "Garside element" ([5]). In this reference, by considering new Garside element as $\delta = a_{nn-1}a_{n-1n-2}\cdots a_{21}$, it has been defined a presentation

$$B_n = gp < a_{ts}\ (n \geq t > s \geq 1) \quad ; \quad a_{ts}a_{rq} = a_{rq}a_{ts}\ ((t-r)(t-q)(s-r)(s-q) > 0)\,,$$
$$a_{ts}a_{sr} = a_{tr}a_{ts} = a_{sr}a_{tr}\ (n \geq t > s > r \geq 1) >,$$

where $a_{ts} = (a_{t-1}a_{t-2}\cdots a_{s+1})a_s(a^{-1}_{s+1}\cdots a^{-1}_{t-2}a^{-1}_{t-1})$ for this braid group B_n (see [5, Proposition 2.1]). By using this specific group with its presentation, Bokut ([10]) obtained a Gröbner-Shirshov basis as in the following theorem. To keep his notation in this theorem, suppose we have an ordering $n \geq t_3 > t_2 > t_1 \geq 1$, let us use the notation (i,j) instead of a_{ij} for $i > j$, and also let us assume that $V_{[k,l]}, V_{2[k,l]}, \cdots, V_{n-1[k,l]}, V'_{2[k,l]}, \cdots, V'_{n-1[k,l]}, \cdots$ denote any words in terms of (i,j) such that $k \geq i > j \geq l$. It is also used the following notations:

$$V_{[t_2-1,t_1]}(t_2,t_1) = (t_2,t_1)V'_{[t_2-1,t_l]}, \quad t_2 > t_1,$$
$$W_{[t_2,t_1]}(t_1,t_0) = (t_1,t_0)W'_{[t_2,t_l]}, \quad t_2 > t_1 > t_0,$$

where

$$V'_{[t_2-1,t_l]} = (V_{[t_2-1,t_1]})\,\big|_{(k,l)\mapsto(k,l) \text{ if } l\neq t_1;\ (k,t_1)\mapsto(t_2,k) \text{ otherwise}}$$

and

$$W'_{[t_2,t_l]} = (W_{[t_2,t_1]})\Big|_{(k,l)\mapsto(k,l) \text{ if } l\neq t_1;\,(k,t_1)\mapsto(k,t_0) \text{ otherwise}},$$

respectively.

Theorem 3.3. [10, Theorem 3.7] *A Gröbner-Shirshov basis $\Re_{B_n}$ of B_n in the Birman-Ko-Lee generators consists of the following relations:*

$$(k,l)(i,j) = (i,j)(k,l), \quad k > l > i > j,$$
$$(k,l)V_{[j-1,1]}(i,j) = (i,j)(k,l)V_{[j-1,1]}, \quad k > i > j > l,$$
$$(t_3,t_2)(t_2,t_1) = (t_2,t_1)(t_3,t_1),$$
$$(t_3,t_1)V_{[t_2-1,1]}(t_3,t_2) = (t_2,t_1)(t_3,t_1)V_{[t_2-1,1]},$$
$$(t,s)V_{[t_2-1,1]}(t_2,t_1)W_{[t_3-1,t_1]}(t_3,t_1) = (t_3,t_2)(t,s)V_{[t_2-1,1]}(t_2,t_1)W'_{[t_3-1,t_1]},$$
$$(t_3,s)V_{[t_2-1,1]}(t_2,t_1)W_{[t_3-1,t_1]}(t_3,t_1) = (t_2,s)(t_3,s)V_{[t_2-1,1]}(t_2,t_1)W'_{[t_3-1,t_1]},$$
$$(2,1)V_{2[2,1]}(3,1)\cdots V_{n-1[n-1,1]}(n,1) = \delta V'_{2[2,1]}\cdots V'_{n-1[n-1,1]},$$
$$(t,s)\delta = \delta(t+1,s+1), \quad (t,s)\delta^{-1} = \delta^{-1}(t-1,s-1), \tag{1}$$
$$\delta\delta^{-1} = 1, \quad \delta^{-1}\delta = 1,$$

where $t > t_3$, $t_2 > s$, and each $t \pm 1$ and $s \pm 1$ in the factors need to be considered by modulo n in the relations given in (1).

In Theorem 3.3, all relations but last two are positive in Birman-Ko-Lee generators. So, by Theorem 2.5, we can say that "*the semigroup of positive braids BB_n^+ in Birman-Ko-Lee generators can be embedded into a group*".

Additionally to previous two different generator type of braid groups, as an example of our main result Theorem 2.5, we can also consider the Gröbner-Shirshov bases of the braid semigroup and the braid group in terms of *Adyan-Thurston generators* (see [22]).

3.1. **Final Remarks. 1)** Each of the groups presented in the papers [11, 12, 13, 14] has actually a *relative* Gröbner-Shirshov basis ([18]) with our condition R_+. Then one can directly say that the initial semigroups given in these references are embeddable into groups.

2) Among these all above braid groups in the specific generators, we further have the braid group in the *Gorin-Lin generators* (see [19, Sections 4 and 5]). In this reference, Bokut et. al obtained Gröbner-Shirshov bases for the braid groups B_3 and B_4 in the Gorin-Lin generators with presentations

$$B_3 = gp\left\langle t_1, t_2, t\ ;\ t^{-1}t_2t = t_2t_1^{-1}, t^{-1}t_1t = t_2 \right\rangle$$

and

$$B_4 = gp\left\langle a, b, t_1, t_2, t\ ;\ t_1t = tt_2, t_2t = tt_2t_1^{-1}, bt = tb, at_1 = t_1b, a = b^{-1}t_2bt_2^{-1} \right\rangle,$$

respectively. Explicitly, the Gröbner-Shirshov basis $\Re_{B_3}$ of B_3 in the Gorin-Lin generators consists of the relations

$$(t_2)^{\pm1}t = t(t_2t_1^{-1})^{\pm1}, \qquad (t_1)^{\pm1} = t(t_2)^{\pm1},$$
$$(t_1)^{\pm1}t^{-1} = t^{-1}(t_2^{-1}t_1)^{\pm1}, \quad (t_2)^{\pm1}t^{-1} = t^{-1}(t_1^{-1})^{\pm1}$$

together with the trivial relations $tt^{-1} = 1 = t^{-1}t$, $t_it_i^{-1} = 1 = t_i^{-1}t_i$ ($i = 1,2$). But as it is seen that the Gröbner-Shirshov basis $\Re_{B_3}$ of B_3 does not satisfy the condition R_+. So we can not use our result (see Theorem 2.5) to check whether the embeddability holds for a braid semigroup in the Gorin-Lin generators. Moreover, if we look at the Gröbner-Shirshov basis $\Re_{B_4}$ of B_4 in the Gorin-Lin generators ([19, pg. 367-368]), then we can easily conclude that Theorem 2.5 cannot be used for checking the embeddability result for the related braid semigroup.

3) It is known that a *Clifford semigroup* can be embedded in a group if and only if it is already a group (see, for instance, [23, 31]). Therefore, for a future project, one can also investigate whether the Clifford semigroups (not with a group approximation) can be an example for Theorem 2.5 under a suitable Gröbner-Shirshov basis. In fact, by this investigation, it can be also answered the same question for *coregular semigroups* since each coregular semigroup is a Clifford semigroup and since coregular semigroups cannot be embedded into a group by using the known methods.

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