Available online at www.sciencedirect.com
ScienceDirect
Chaos, Solitons and Fractals 40 (2009) 106–110
CHAOS
SOLITONS & FRACTALS
www.elsevier.com/locate/chaos

On the n-transitivity of the group of Möbius transformations on $\mathbb{C}_\infty$

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Accepted 10 July 2007

Abstract

Möbius transformations generate the conformal group in the plane and have been used in neural networks and conformal field theory. Some invariant characteristic properties of Möbius transformations such as the invariance of cross-ratio of four distinct points on the extended complex plane $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ under a Möbius transformation, have many applications. We consider the geometric interpretation of the notion of n-transitivity of the group of Möbius transformations on the extended complex plane $\mathbb{C}_\infty$. We see that this notion is closely related to the invariant characteristic properties of Möbius transformations and the notion of cross-ratio.
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1. Introduction

A Möbius transformation T has the form

$$T(z) = \frac{az+b}{cz+d}; \quad a,b,c,d \in \mathbb{C} \quad \text{and} \quad ad - bc \neq 0.$$

These transformations form a group under composition. We denote this group by $\mathfrak{M}$. It is well-known that Möbius transformations map circles to circles (where straight lines are considered to be circles through ∞). Conversely, that any circle preserving meromorphic map of the extended complex plane onto itself is a Möbius transformation, (see [8,20]). Therefore the principle of circle transformation is an invariant characteristic property of Möbius transformations.

Recently, several new invariant characteristic properties of Möbius transformations have been given (see [2,3,9–12,21–24]). These results require some known results from geometry together with well-known properties of Möbius transformations such as the invariance of cross-ratio of four distinct points on $\mathbb{C}_\infty$ under a Möbius transformation. Furthermore, some new geometric concepts were introduced and used for this purpose. For example, the concepts of "k-Apollonius quadrilateral" and " k-Apollonius $2n$-gon" were introduced (see [21,24] for more details and examples, respectively).

Möbius transformations have been used in general neural networks, signal processing (see [6,14]), conformal field theory and Cantorian $E^{(\infty)}$ theory (see [7,15–19]). The set of all Möbius transformations of the form

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doi:10.1016/j.chaos.2007.07.024

$$T(z) = \frac{az+b}{cz+d}, \tag{1}$$

where a, b, c, d are integers with $ad - bc = 1$, form a subgroup of $\mathfrak{M}$ and is called the modular group. The two transformations $R(z) = -\frac{1}{z}$ and $S(z) = z + 1$ generate the entire modular group. In the above studies, modular group plays an important role. In [14], it was shown that both a nonlinear activation function of a neuron and a first order all-pass filter section can be considered as Möbius transformations. Some inherent properties of neural networks, such as fixed points and invertibility, and group delay properties of cascaded all-pass filters, were shown to be the consequence of their Möbius representations (for more details see [6,14]). In [15], El Naschie showed the link between the fixed points of the modular groups of the vacuum and the golden mean $\phi = \frac{1}{1+\phi} = \frac{\sqrt{5}-1}{2}$ of $E^{(\infty)}$ spacetime by analytical continuation of a Möbius transformation. In (1), if we assume that $a - d = b = c$ and $d > 0$, then it can be easily seen that T has two distinct fixed points namely,

$$\frac{1}{\phi} = 1.6180339887\ldots \quad \text{and} \quad -\phi = -0,6180339887\ldots.$$

In [19], some connections between string theory and $E^{(\infty)}$ theory mediated by transformations of the modular group were discussed. It was studied the behaviour of certain quantum probabilities under global diffeomorphisms generated by transformations of the modular group. To do that it is sufficient to consider the generators of the modular group. $\phi = \frac{\sqrt{5}-1}{2}$ is the $d_c^{(0)}$ of $E^{(\infty)}$ theory. In [18], it was shown that Klein modular curve is the holographic boundary of $E^{(\infty)}$ Cantorian theory. For more details see [7,15–19]. For the usage of the notion of cross-ratio, one can consult [5,25].

In this paper we deal with the geometric interpretation of the notion of n -transitivity of the group $\mathfrak{M}$ on $\mathbb{C}_\infty$. We show that this notion is closely related to the invariant characteristic properties of Möbius transformations and the notion of cross-ratio.

2. Main results

At first, we consider the case $n = 4$. If z_1, z_2, z_3, z_4 are four distinct points in $\mathbb{C}_\infty$, the cross-ratio and the absolute cross-ratio of these points are defined by

$$[z_1, z_2; z_3, z_4] = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}, \quad |z_1, z_2; z_3, z_4| = \frac{|z_1 - z_2|.|z_3 - z_4|}{|z_2 - z_3|.|z_4 - z_1|}, \tag{2}$$

respectively. Möbius transformations preserve cross-ratios and absolute cross-ratios. The connection between Möbius transformations, cross-ratios and the preservation of circles is well-known. It is well-known that $\mathfrak{M}$ acts 3-transitively but not 4-transitively on $\mathbb{C}_\infty$. The following theorem is also well-known:

Theorem 2.1 [13]. *Let* (z_1, z_2, z_3, z_4) *and* (w_1, w_2, w_3, w_4) *be* 4*-tuples of distinct elements in* $\mathbb{C}_\infty$. *Then there exists some Möbius transformation* T *with* $T(z_j) = w_j$ $(j = 1, 2, 3, 4)$ *if and only if* $[z_1, z_2; z_3, z_4] = [w_1, w_2; w_3, w_4]$.

Note that the Möbius transformation T in Theorem 2.1 is unique.

In [11], Haruki and Rassias introduced the concept of "*Apollonius quadrilateral*" to give a new characterization of Möbius transformations. The notion of "*Apollonius quadrilateral*" is closely related to the notion of "cross-ratio". This connection was not mentioned explicitly in [11]. This connection was stated in [1]. Afterwards, in [21], Niamsup generalized the notion of Apollonius quadrilateral to the notion of k-Apollonius quadrilateral where $k > 0$. Then, by means of this definition, a new invariant characteristic property of Möbius transformations was given. We recall this definition from [21].

Definition 2.2. Let $ABCD$ be an arbitrary quadrilateral (not necessarily simple) on $\mathbb{C}$. If $\overline{AB} \cdot \overline{CD} = k(\overline{BC} \cdot \overline{DA})$ holds, then $ABCD$ is said to be a k-Apollonius quadrilateral.

Property 1. *Suppose that* $w = f(z)$ *is analytic and univalent in a nonempty domain* R *of the* z*-plane. Let* $ABCD$ *be an arbitrary* k*-Apollonius quadrilateral contained in* R. *If we set* $A' = f(A)$, $B' = f(B)$, $C' = f(C)$, $D' = f(D)$, *then* $A'B'C'D'$ *is also a* k*-Apollonius quadrilateral.*

Theorem 2.3 [21]. *The function* $w = f(z)$ *satisfies* Property 1 *iff* $w = f(z)$ *is a Möbius transformation.*

Clearly, four distinct points A, B, C and D on $\mathbb{C}$ form the vertices of a k-Apollonius quadrilateral if and only if $|A, B; C, D| = k$. So we get a Möbius transformation sends one k-Apollonius quadrilateral to another, and the "only if part" of Theorem 2.3, when stated in terms of absolute cross-ratio, reads as follows.

Theorem 2.4. *Suppose that f is meromorphic in some domain in* $\mathbb{C}$, *and that for every A, B, C and D*, $|A,B;C,D| = k$ *implies* $|f(A),f(B);f(C),f(D)| = k$. *Then f is a Möbius transformation.*

Theorem 2.4 is a generalization of Theorem A in [1].

Now we extend the notion of k-Apollonius quadrilateral. In Definition 2.2, we permit all of the points A, B, C and D or any triple of them to be on the same straight line or one of these points to be ∞, and we call such k-Apollonius quadrilaterals as degenerate k-Apollonius quadrilaterals. For example a line segment or a triangle will represent a degenerate k-Apollonius quadrilateral whose vertices are co-linear or whose tree vertices are co-linear. From now on we use the term "k-Apollonius quadrilateral" to mean both of the degenerate or non-degenerate k-Apollonius quadrilaterals. This extension allows us to do following observation.

There is a connection between the notions of k-Apollonius quadrilateral and 4-transitivity of the group $\mathfrak{M}$ on $\mathbb{C}_\infty$. Firstly, we note that any distinct four points on the extended complex plane form the vertices of a k-Apollonius quadrilateral. Indeed, let the points z_1, z_2, z_3 and z_4 be any distinct four points on the complex plane. If we write

$$\frac{|z_1 - z_2||z_3 - z_4|}{|z_2 - z_3||z_4 - z_1|} = |\lambda|,$$

where $\lambda = [z_1, z_2; z_3, z_4]$, we get

$$|z_1 - z_2||z_3 - z_4| = |\lambda||z_2 - z_3||z_4 - z_1|.$$

That is, the points z_1, z_2, z_3 and z_4 form the vertices of the k-Apollonius quadrilateral where $k = |\lambda|$. Combining these facts with Theorem 2.1, we get the following theorem.

Theorem 2.5. *Let* (z_1, z_2, z_3, z_4) *and* (w_1, w_2, w_3, w_4) *be 4-tuples of distinct elements in* $\mathbb{C}_\infty$. *If there exists some Möbius transformation T with* $T(z_i) = w_i$ $(i = 1, 2, 3, 4)$, *then the points* z_1, z_2, z_3, z_4 *form of the vertices of a k-Apollonius quadrilateral and also the points* w_1, w_2, w_3, w_4 *form of the vertices of another k-Apollonius quadrilateral, where* $k = |z_1, z_2; z_3, z_4| = |w_1, w_2; w_3, w_4|$.

As a generalization of the notion of k-Apollonius quadrilateral, Samaris gave the following definition of k-Apollonius $2n$-gon, [24].

Definition 2.6. A $2n$-gon (not necessarily simple) on the complex plane is called k-Apollonius if for the consecutive vertices $z_1, z_2, \ldots, z_{2n} \in \mathbb{C}$, the following condition holds

$$A(z_1, z_2, \ldots, z_{2n}) = k, \tag{3}$$

where

$$A(z_1, z_2, \ldots, z_{2n}) = \frac{|(z_1 - z_2)(z_3 - z_4)\ldots(z_{2n-1} - z_{2n})|}{|(z_2 - z_3)(z_4 - z_5)\ldots(z_{2n-2} - z_{2n-1})(z_{2n} - z_1)|}. \tag{4}$$

In [24], the following invariant characteristic property of Möbius transformations was given.

Theorem 2.7. *If f is an analytic univalent function on an open region* $\triangle$ *then the following propositions are equivalent*:

(i) *f is a Möbius transformation.*
(ii) *There is* $k > 0$ *such that* $A(f(z_1), f(z_2), \ldots, f(z_{2n})) = k$, *for every* $z_1, z_2, \ldots, z_{2n} \in \triangle$ *with* $A(z_1, z_2, \ldots, z_{2n}) = k$.

Again we extend the notion of k-Apollonius $2n$-gon in a similar manner. In Definition 2.6, we permit one of the points z_i to be infinity, or any number of them to be on the same straight line. We call such a k -Apollonius $2n$-gon as degenerate k-Apollonius $2n$-gon. We use the term "k-Apollonius $2n$-gon" to mean both of the degenerate or non-degenerate k-Apollonius $2n$-gons.

There is an interesting connection between the notion of k-Apollonius $2m$ -gon and $2m$-transitivity ($m \geqslant 3$) of the group $\mathfrak{M}$ on $\mathbb{C}_\infty$ as in the case k-Apollonius quadrilateral and 4-transitivity for $m = 2$. We have the following theorem:

Theorem 2.8. *Let* $(z_1, \ldots, z_{2m})$ *and* $(w_1, \ldots, w_{2m})$ *be 2m-tuples of distinct elements in* $\mathbb{C}_\infty$, $(m \geqslant 3)$. *If there exists some Möbius transformation T with* $T(z_i) = w_i$ $(1 \leqslant i \leqslant 2m)$, *then the points* $z_1, \ldots, z_{2m}$ *form the vertices of a k-Apollonius 2m-gon and also the points* $w_1, \ldots, w_{2m}$ *form the vertices of another k-Apollonius 2m-gon where* $k = |\lambda_1| \cdot |\lambda_2| \cdot \ldots |\lambda_{m-1}|$ *and* $\lambda_i = [z_1, z_{2i}; z_{2i+1}, z_{2i+2}]$, $(1 \leqslant i \leqslant m - 1)$.

Proof. Let $[z_1, z_2; z_3, z_4] = \lambda_1$. Then we have

$$\frac{|z_1 - z_2| \cdot |z_3 - z_4|}{|z_2 - z_3| \cdot |z_4 - z_1|} = |\lambda_1|. \tag{5}$$

If we take $[z_1, z_4; z_5, z_6] = \lambda_2$, we get

$$\frac{|z_1 - z_4| \cdot |z_5 - z_6|}{|z_4 - z_5| \cdot |z_6 - z_1|} = |\lambda_2| \tag{6}$$

and hence

$$|\lambda_1| \cdot |\lambda_2| = \frac{|z_1 - z_2| \cdot |z_3 - z_4| \cdot |z_5 - z_6|}{|z_2 - z_3| \cdot |z_4 - z_5| \cdot |z_6 - z_1|}. \tag{7}$$

Repeating this process, if we take $[z_1, z_{2m-2}; z_{2m-1}, z_{2m}] = \lambda_{m-1}$, we have

$$|\lambda_1| \cdot \ldots \cdot |\lambda_{m-1}| = \frac{|z_1 - z_2| \cdot |z_3 - z_4| \cdots |z_{2m-1} - z_{2m}|}{|z_2 - z_3| \cdot |z_4 - z_5| \ldots |z_{2m-2} - z_{2m-1}| \cdot |z_{2m} - z_1|} = A(z_1, z_2, \ldots, z_{2m}).$$

By Definition 2.6, the points $z_1, \ldots, z_{2m}$ form the vertices of the k-Apollonius $2m$-gon where $k = |\lambda_1| \cdot |\lambda_2| \ldots |\lambda_{m-1}|$ and $\lambda_i = [z_1, z_{2i}; z_{2i+1}, z_{2i+2}]$, $(1 \leqslant i \leqslant m-1)$. If there exists some Möbius transformation T with $T(z_i) = w_i$ $(1 \leqslant i \leqslant 2m)$, then by Theorem 2.7, we have $A(z_1, z_2, \ldots, z_{2m}) = A(w_1, w_2, \ldots, w_{2m})$, that is, the points $w_1, \ldots, w_{2m}$ form the vertices of another k-Apollonius $2m$-gon where $k = |\lambda_1| \cdot |\lambda_2| \cdot \ldots \cdot |\lambda_{m-1}|$. □

Finally we give a necessary and sufficient condition for the n -transitivity $(n \geqslant 5)$ of the group $\mathfrak{M}$ on $\mathbb{C}_\infty$ without any restriction on n.

Theorem 2.9. *Let $(z_1, \ldots, z_n)$ and $(w_1, \ldots, w_n)$ be n-tuples of distinct elements in $\mathbb{C}_\infty$, $(n \geqslant 5)$. Then there exists some Möbius transformation T with $T(z_i) = w_i$ $(1 \leqslant i \leqslant n)$ if and only if the cross-ratio of any four of the points z_i $(1 \leqslant i \leqslant n)$ is equal to the cross-ratio of the corresponding points w_i.*

Proof. Suppose that $T(z_i) = w_i$ $(1 \leqslant i \leqslant n)$. Let z_j, z_k, z_l, z_m be the any four of the points z_i $(1 \leqslant i, j, k, l, m \leqslant n)$ and w_j, w_k, w_l, w_m be the corresponding ones. Let

$$U(z) = \frac{(z - w_k)(w_l - w_m)}{(w_k - w_l)(w_m - z)}$$

be the unique Möbius transformation sending w_k, w_l, w_m to 0, 1, ∞, respectively. We have $U(w_j) = [w_j, w_k; w_l, w_m]$. Then UT is a Möbius transformation sending z_k, z_l, z_m to 0, 1, ∞, respectively. Therefore UT is unique and we get

$$[z_j, z_k; z_l, z_m] = UT(z_j) = U(w_j) = [w_j, w_k; w_l, w_m].$$

Conversely, if T is the Möbius transformation mapping z_i to w_i $(1 \leqslant i \leqslant 4)$ and S is the Möbius transformation mapping z_i to w_i for $i = 5, 2, 3, 4$ (by Theorem 2.1, T and S exist because of the condition on the equality of the cross-ratios), then $S = T$ because both Möbius transformations map z_i to w_i for $i = 2, 3, 4$, and a Möbius transformation is determined by the image of three points. The same line of argument works for other z_i and w_i pairs with $i \geqslant 6$. Therefore, T is the unique Möbius transformation which map z_i to w_i $(1 \leqslant i \leqslant n)$. □

We can give the geometric interpretation of Theorem 2.9 as follows:

Theorem 2.10. *Let $(z_1, \ldots, z_n)$ and $(w_1, \ldots, w_n)$ be n-tuples of distinct elements in $\mathbb{C}_\infty$, $(n \geqslant 5)$. If there exists some Möbius transformation T with $T(z_i) = w_i$ $(1 \leqslant i \leqslant n)$, then any four of the points z_i $(1 \leqslant i \leqslant n)$ form the vertices of a k-Apollonius quadrilateral and the corresponding points w_j also form of the vertices of another k-Apollonius quadrilateral where k is the equal absolute cross-ratios of these points z_i and w_i.*

3. Conclusions

Möbius transformations have many applications in mathematical physics. In this paper, we have obtained the geometric interpretation of the notion of n -transitivity of the group of Möbius transformations on the extended complex plane. We have seen that this notion is closely related to the invariant characteristic properties of Möbius transformations and the notion of cross-ratio.

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