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ORIGINAL PAPER

Fractional optimal control of a 2-dimensional distributed system using eigenfunctions

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Abstract This paper presents an eigenfunctions expansion based scheme for Fractional Optimal Control (FOC) of a 2-dimensional distributed system. The fractional derivative is defined in the Riemann–Liouville sense. The performance index of a FOC problem is considered as a function of both state and control variables, and the dynamic constraints are expressed by a Partial Fractional Differential Equation (PFDE) containing two space parameters and one time parameter. Eigenfunctions are used to eliminate the terms containing space parameters and to define the problem in terms of a set of generalized state and control variables. For numerical computation Grünwald–Letnikov approximation is used. A direct numerical technique is proposed to obtain the state and the control variables. For a linear case, the numerical technique results into a set of algebraic equations which can be solved using a direct or an iterative scheme. The problem is solved for different number of eigenfunctions and time discretization. Numerical results show that only a few eigenfunctions are sufficient to obtain good results, and the solutions converge as the size of the time step is reduced.

Keywords Eigenfunction · Fractional derivative · Fractional optimal control · Grünwald–Letnikov approximation · Riemann–Liouville derivative · Two-dimensional distributed system

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1 Introduction

Fractional calculus deals with the generalization of differentiation and integration of noninteger orders. In recent years, it has played a significant role in physics, chemistry, biology, electronics, and control theory. Extensive treatment and various applications of the fractional calculus are discussed in [1–9]. It has been demonstrated that Fractional Order Differential Equations (FODEs) model dynamic systems and processes more accurately than integer order differential equations do, and fractional controllers perform better than integer order controllers (see, [1, 6, 7, 10–17]).

Oustaloup [18] introduced fractional order controls for dynamic systems and applied them to control a car suspension and a flexible-transmission-hydraulic actuator. It was demonstrated that the CRONE (*Commande Robuste d'Ordre Non-Entier*) method out perform the PID control. References [6, 19] demonstrate that a fractional $PI^{\lambda}D^{\mu}$ controller performs better than classical PID controller when used for control of fractional order systems. Fractional PD^{δ} and other controllers have been suggested in [10, 20]. More recently, new fractional order controllers have been developed and applied in robotics control (see, [14, 16, 21, 22]). Applications of fractional order controllers to viscoelastically damped structures could be found in [23].

The papers cited should be sufficient to emphasize the fact that fractional controllers are becoming popular. However, these papers do not develop the field of fractional optimal control (FOC). Although a significant amount of work has been done in the area of Integer Order Optimal Controls (IOOCs), very little work has been published in the area of optimal control of Fractional Dynamical Systems (FDSs), particularly in FOC of distributed systems. Like the formulation of an IOOC problem, the roots of the formulations of Fractional Order Optimal Control Problems (FOCPs) lie in variational calculus. Excellent books, review articles, and papers are available on Integer Order Variational Calculus (IOVC) (see, [24–27]). However, they are limited to integer order systems.

Lately, some authors have presented theories and analytical and numerical schemes for FOCPs. In [28], the Euler–Lagrange equations for fractional calculus of variations which has been the basis for several FOCPs developed later. Reference [29] presents a general formulation and a numerical scheme for FOCPs. A general scheme for stochastic analysis of FOCPs is presented in [30]. A formulation for FOCPs is defined in terms of Caputo fractional derivatives in [31, 32] and in terms of Riemann–Liouville (R–L) fractional derivatives in [33]. Reference [34] presents an eigenfunction expansion approach for an FOC formulation of a one dimensional distributed systems in terms of Caputo fractional derivatives. The formulation leads to an infinite dimensional FOCP. However, the resulting differential equations can be grouped into infinite sets, each of which could be solved independently. Several authors have recently considered solutions of fractional diffusion-wave equations defined in multi-dimensional space (e.g., see [35, 36] and the references therein). However, these papers focus on the response of the system, and they do not consider FOC.

This paper presents a formulation and some numerical results for FOC of a two- dimensional distributed system. The fractional derivative is defined in the Riemann–Liouville sense. The performance index of a FOCP is considered as a function of both state and control variables, and the dynamic constraints are expressed by a fractional diffusion-wave equation containing two space parameters and one time parameter. The formulation uses an eigenfunction approach to transform the continuum problem to a problem in countable infinite dimension and a Hamiltonian approach to obtain the fractional differential equations of the system (see, [33, 34]). For numerical computation, the Grünwald–Letnikov (G–L) approximation is used. It is a simple but effective method for evaluation of fractional-order derivatives. This approach is based on the fact that in the limit, for a wide class of functions appearing in real and engineering applications, the R–L and the G–L definitions are equivalent. This allows one to use the R–L definition during problem formulation, and then turn to the G–L definition for obtaining the numerical solution (see, [6]). The problem is solved for different number of eigenfunctions and time discretization. The formulation is very similar to the formulation presented in [34] with three exceptions: (1) The formulation in [34] considers the Caputo fractional derivatives whereas this research considers the Riemann–Liouville fractional derivatives, (2) Reference [34] converts the resulting equations to Volterra type integral equations, whereas this paper uses a direct numerical scheme to solve the resulting equations, and (3) Reference [34] considers a 1-dimensional distributed system, whereas this paper considers a 2-dimensional distributed system.

The paper is organized as follows. In Sect. 2, the R–L fractional derivative, the G–L approximation, and an FOCP are briefly reviewed. In Sect. 3, the FOC of a Two Dimensional Distributed System (TDDS) is formulated using the approach presented in [33]. Section 4 presents numerical results for the TDDS to show the effectiveness of the approach. Finally, Sect. 5 is dedicated to conclusions.

2 Fractional optimal control problem

Many definitions have been given of a fractional derivative, which include Riemann–Liouville, Grünwald–Letnikov, Weyl, Caputo, Marchaud, and Riesz fractional derivative (see, [1, 2, 6, 37]). We will formulate the problem in terms of the left and right Riemann–Liouville fractional derivatives which are given as

The left Riemann–Liouville fractional derivative:

$$
{}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \times \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\, d\tau. \tag{1}
$$

The right Riemann–Liouville fractional derivative:

$$
{}_tD_b^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dt}\right)^n \times \int_t^b (\tau-t)^{n-\alpha-1} f(\tau)\, d\tau \tag{2}
$$

where $f(.)$ is a time dependent function, $\Gamma(.)$ is the gamma function, and α is the order of the derivative such that $n-1<\alpha<n$, n is an integer number. These derivatives will be denoted as the LRLFD and the RRLFD, respectively. When α is an integer the left (forward) and the right (backward) derivatives are replaced with D and $-D$, respectively, where D is an ordinary differential operator. Note that in the literature the Riemann–Liouville fractional derivative generally means the LRLFD.

Using these definitions, the FOCP can be defined as follows: Find the optimal control $u_{ij}(t)$ that minimizes the performance index

$$
J(u_{ij}) = \int_0^1 F(x_{ij}, u_{ij}, t)\, dt \tag{3}
$$

subject to the system dynamic constraints

$$
{}_aD_t^{\alpha} x_{ij} = G(x_{ij}, u_{ij}, t) \tag{4}
$$

and the initial condition

$$
x_{ij}(0) = x_{ij0}, \tag{5}
$$

where t represents time, $x_{ij}(t)$ and $u_{ij}(t)$ are the state and the control variables, respectively, and F and G are two arbitrary functions. In defining the above problem, subscripts ij are not necessary. However, our later derivations will require these subscripts. Therefore, for consistency, they are included here, also. Equation (3) may include additional terms containing state variables at the end points. When $\alpha=1$, the above problem reduces to a standard optimal control problem. Here, we take $0<\alpha<1$. It is assumed that $x_{ij}(t)$, $u_{ij}(t)$, and $G(x_{ij}, u_{ij}, t)$ are all scalar functions. These assumptions are made for simplicity. The same procedure could be followed if the upper limit of integration is different from 1, α is greater than 1, and/or $x_{ij}(t)$, $u_{ij}(t)$, and $G(x_{ij}, u_{ij}, t)$ are vector functions. However, in the case of $\alpha>1$, additional initial conditions may be necessary. In optimal control formulations, traditionally the differential equations governing the dynamics of the system are written in state space form, in which case, the order of the derivatives turns out to be less than 1. For this reason, we consider $0<\alpha<1$.

To find the optimal control, we define a modified performance index as

$$
J(u_{ij}) = \int_0^1 \left[H(x_{ij}, u_{ij}, \lambda_{ij}, t) - \lambda_{ij}\, {}_0D_t^{\alpha} x_{ij}(t)\right] dt \tag{6}
$$

where $H(x_{ij}, u_{ij}, \lambda_{ij}, t)$ is the Hamiltonian of the system defined as

$$
H(x_{ij}, u_{ij}, \lambda_{ij}, t) = F(x_{ij}, u_{ij}, t) + \lambda_{ij} G(x_{ij}, u_{ij}, t). \tag{7}
$$

Here λ_{ij} is the Lagrange multiplier. Using the approach presented in [33], the necessary conditions for the optimal control are given as

$$
{}_tD_1^{\alpha} \lambda_{ij} = \frac{\partial F}{\partial x_{ij}} + \lambda_{ij} \frac{\partial G}{\partial x_{ij}}, \tag{8}
$$

$$
\frac{\partial F}{\partial u_{ij}} + \lambda_{ij} \frac{\partial G}{\partial u_{ij}} = 0, \tag{9}
$$

$$
{}_0D_t^{\alpha} x_{ij} = G(x_{ij}, u_{ij}, t) \tag{10}
$$

and

$$
\lambda_{ij}(1) = 0. \tag{11}
$$

Another approach to obtain these conditions is presented in [29].

Equations (8)–(11) represent the Euler–Lagrange equations for the FOCP defined by (3)–(5). They are very similar to the Euler–Lagrange equations for classical optimal control problems, except that the resulting differential equations contain the left and the right fractional derivatives. This indicates that the solution of optimal control problems requires knowledge of not only forward derivatives but also backward derivatives to account for end conditions. This issue is not discussed in classical optimal control theory.

In the next section, a formulation for an FOC of a 2-D distributed system will be presented, and an eigenfunctions expansion method will be used to reduce this formulation into a set of FOCPs in which each FOCP can be solved independently.

3 Formulation of fractional optimal control of a 2-dimensional distributed system

We consider the following problem: Find the control $u(\xi,\eta,t)$ which minimizes the cost functional

$$J(u)=\frac{1}{2}\int_0^1\int_0^L\int_0^L\big[Q'x^2(\xi,\eta,t) + R'u^2(\xi,\eta,t)\big]\,d\xi\,d\eta\,dt \tag{12}$$

subjected to the system dynamic constraints

$${}_0D_t^\alpha x(\xi,\eta,t)=\frac{\partial^2 x(\xi,\eta,t)}{\partial\xi^2}+\frac{\partial^2 x(\xi,\eta,t)}{\partial\eta^2} + u(\xi,\eta,t), \tag{13}$$

the initial condition

$$x(\xi,\eta,0)=x_0(\xi,\eta), \tag{14}$$

and the boundary conditions

$$\frac{\partial x(0,\eta,t)}{\partial\xi}=\frac{\partial x(L,\eta,t)}{\partial\xi}=\frac{\partial x(\xi,0,t)}{\partial\eta} =\frac{\partial x(\xi,L,t)}{\partial\eta}=0, \tag{15}$$

where $x(\xi,\eta,t)$ and $u(\xi,\eta,t)$ are the state and the control functions which depend on three parameters $(\xi\ ,\eta\ ,t)$, ${}_0D_t^\alpha x(\xi,\eta,t)$ represents the R–L fractional derivative of $x(\xi,\eta,t)$ of order $\alpha>0$ with respect to time t, $(\xi,\ \eta)\in[0,\ L]\times[0,\ L]$ are the space parameters, and Q' and R' are two arbitrary functions which may depend on time. Note that the partial derivative symbol is used here to emphasize the fact that $x(\xi,\eta,t)$ and $u(\xi,\eta,t)$ depend in addition to t on ξ and η also. The upper limit for time t is taken as 1 for convenience. This limit could be any positive number. The formulation developed here is not limited to this system, but it can also be applied to other distributed systems.

Using the eigenfunctions $\phi_{ij}(\xi,\eta)$, $i,j=0,1,2,\dots,\infty$, functions $x(\xi,\eta,t)$ and $u(\xi,\eta,t)$ can be written as

$$x(\xi,\eta,t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}x_{ij}(t)\phi_{ij}(\xi,\eta) \tag{16}$$

and

$$u(\xi,\eta,t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}u_{ij}(t)\phi_{ij}(\xi,\eta) \tag{17}$$

where $x_{ij}(t)$ and $u_{ij}(t)$ are the state and the control eigencoordinates. Using the method of separation of variables, it could be demonstrated that the eigenfunctions for this problem are given as

$$\phi_{ij}(\xi,\eta)=\cos\left(i\pi\frac{\xi}{L}\right)\cos\left(j\pi\frac{\eta}{L}\right),\quad i,j=0,1,2,\dots,\infty. \tag{18}$$

Using direct calculations, it can be demonstrated that in most applications, the terms associated with higher order eigenvalues do not contribute much. Hence, for computational purposes, one needs to consider only a finite number of terms. Furthermore, the maximum limits considered for i and j need not be the same. However, for simplicity, we shall take m as the upper limits for both i and j.

Substituting (16) and (17) into (12), we get

$$J=\frac{L^2}{2}\int_0^1\Bigg[Q'\Bigg(x_{00}^2+\sum_{i=1}^{m}\frac{1}{2}x_{i0}^2+\sum_{j=1}^{m}\frac{1}{2}x_{0j}^2 +\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{1}{4}x_{ij}^2\Bigg) + R'\Bigg(u_{00}^2+\sum_{i=1}^{m}\frac{1}{2}u_{i0}^2+\sum_{j=1}^{m}\frac{1}{2}u_{0j}^2 +\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{1}{4}u_{ij}^2\Bigg)\Bigg]dt. \tag{19}$$

Substituting (16) and (17) into (13), and equating the coefficients of $\phi_{ij}(\xi,\eta)$, we get

$${}_0D_t^\alpha x_{ij}(t)=-\left\{\left(\frac{i\pi}{L}\right)^2+\left(\frac{j\pi}{L}\right)^2\right\}x_{ij}(t)+u_{ij}(t),\quad i,j=0,1,\dots,m. \tag{20}$$

Finally, substituting (16) into (14), multiplying both sides by $\cos(i\pi\frac{\xi}{L})\cos(j\pi\frac{\eta}{L})$, and integrating both ξ, η from 0 to L, we get

$$x_{ij}(0) = x_{ij0} = \frac{1}{L^2}\begin{cases} \int_0^L\int_0^L x_0(\xi,\eta)\,d\xi\,d\eta & i=0,\ j=0 \\ 2\int_0^L\int_0^L x_0(\xi,\eta) \times\cos\left(\frac{j\pi\eta}{L}\right)d\xi d\eta & i=0,\ j>0 \\ 2\int_0^L\int_0^L x_0(\xi,\eta) \times\cos\left(\frac{i\pi\xi}{L}\right)d\xi d\eta & j=0,\ i>0 \\ 4\int_0^L\int_0^L x_0(\xi,\eta) \times\cos\left(\frac{i\pi\xi}{L}\right) \times\cos\left(\frac{j\pi\eta}{L}\right)d\xi\,d\eta & i>0,\ j>0. \end{cases} \tag{21}$$

Using the above approximations, the Hamiltonian for this system can be defined as

$$H = \frac{L^2}{2}\left[Q'\left(x_{00}^2 + \sum_{i=1}^m \frac{1}{2}x_{i0}^2 + \sum_{j=1}^m \frac{1}{2}x_{0j}^2 + \sum_{i=1}^m\sum_{j=1}^m \frac{1}{4}x_{ij}^2\right) + R'\left(u_{00}^2 + \sum_{i=1}^m \frac{1}{2}u_{i0}^2 + \sum_{j=1}^m \frac{1}{2}u_{0j}^2 + \sum_{i=1}^m\sum_{j=1}^m \frac{1}{4}u_{ij}^2\right)\right] + \sum_{i=1}^m\sum_{j=1}^m \lambda_{ij}\left[-\left\{\left(\frac{i\pi}{L}\right)^2 + \left(\frac{j\pi}{L}\right)^2\right\}x_{ij}(t) + u_{ij}(t)\right]. \tag{22}$$

The necessary conditions for optimality of this system are given as [33]

$${}_0D_t^\alpha x_{ij}(t) = \frac{\partial H}{\partial \lambda_{ij}}, \tag{23}$$

$$\frac{\partial H}{\partial u_{ij}} = 0, \tag{24}$$

$${}_0D_t^\alpha \lambda_{ij}(t) = \frac{\partial H}{\partial x_{ij}}, \tag{25}$$

$$\lambda_{ij}(1) = 0, \quad i,j = 0,1,\ldots,m. \tag{26}$$

Equations (19) to (21) are very similar to (3) to (5). They are also a finite dimension approximation of (12) to (15).

Using (22) to (25), the necessary conditions for fractional optimal control can be found. For $i=j=0$, they are given as

$$\begin{aligned} &L^2Q'x_{00}(t) - {}_tD_1^\alpha\lambda_{00}(t) = 0, \\ &L^2R'u_{00}(t) - \lambda_{00}(t) = 0, \\ &u_{00}(t) - {}_0D_t^\alpha x_{00}(t) = 0, \end{aligned} \tag{27}$$

otherwise, they are given as

$$\frac{L^2}{4}Q'x_{ij}(t) - \lambda_{ij}(t)\left\{\left(\frac{i\pi}{L}\right)^2 + \left(\frac{j\pi}{L}\right)^2\right\} - {}_tD_1^\alpha\lambda_{ij}(t) = 0, \tag{28}$$

$$\frac{L^2}{4}R'u_{ij}(t) + \lambda_{ij}(t) = 0, \tag{29}$$

$$-\left\{\left(\frac{i\pi}{L}\right)^2 + \left(\frac{j\pi}{L}\right)^2\right\}x_{ij}(t) + u_{ij}(t) - {}_0D_t^\alpha x_{ij}(t) = 0. \tag{30}$$

Substituting $\lambda_{ij}(t)$ from (29) into (28), and after rearranging the terms, we can obtain the differential equations as

$${}_tD_1^\alpha u_{ij}(t) = -\frac{Q'}{R'}x_{ij}(t) - u_{ij}(t)\left\{\left(\frac{i\pi}{L}\right)^2 + \left(\frac{j\pi}{L}\right)^2\right\}, \quad i,j = 0,1,\ldots,m. \tag{31}$$

Using (21), (26), and (28), we obtain the terminal condition as

$$x_{ij}(0) = x_{ij0} \tag{32}$$

and

$$u_{ij}(1) = 0. \tag{33}$$

Note that (30) and (31) depend only on $x_{ij}(t)$ and $u_{ij}(t)$, and they are decoupled from other variables.

Therefore, these two equations along with the end conditions (32) and (33) can be solved using Grünwald–Letnikov approximation. For $\alpha = 1$, (30) and (31) lead to

$$\dot{x}_{ij}(t) = -\left\{\left(\frac{i\pi}{L}\right)^2 + \left(\frac{j\pi}{L}\right)^2\right\}x_{ij}(t) + u_{ij}(t), \quad (34)$$

$$\dot{u}_{ij}(t) = \frac{Q'}{R'}x_{ij}(t) + u_{ij}(t)\left\{\left(\frac{i\pi}{L}\right)^2 + \left(\frac{j\pi}{L}\right)^2\right\}. \quad (35)$$

A closed form solution for this set of equations is given in the Appendix. This solution will be used to demonstrate that in the limit, when α approaches to 1, the numerical approach and the analytical solution overlap.

4 Numerical results

To solve the FOCP we use direct numerical scheme which is proposed in [33]. According to this scheme, the entire time domain is divided into N equal subdomains, and the nodes are labeled as $0, 1, \ldots, N$. For the present case, the size of each subdomain is $h = \frac{1}{N}$, and the time at node j is $t_j = jh$.

Consider the following fractional differential equations, correspond to (20) and (31)

$$_0D_t^{\alpha}x = ax + bu, \quad (36)$$

$$_tD_1^{\alpha}u = cx + du. \quad (37)$$

These equations can be approximated at node M by using Grünwald–Letnikov approximation of the left and right RLFDs as

$$_0D_t^{\alpha}x = \frac{1}{h^{\alpha}}\sum_{j=0}^{M} w_j^{(\alpha)}x(hM - jh), \quad (38)$$

$$_tD_1^{\alpha}u = \frac{1}{h^{\alpha}}\sum_{j=0}^{N-M} w_j^{(\alpha)}u(hM + jh), \quad (39)$$

where

$$w_0^{\alpha} = 1, \qquad w_j^{\alpha} = \left(1 - \frac{\alpha+1}{j}\right)w_{(j-1)}^{\alpha}. \quad (40)$$

Thus, the above equations reduce to

$$\frac{1}{h^{\alpha}}\sum_{j=0}^{M} w_j^{(\alpha)}x(hM - jh) = ax(Mh) + bu(Mh), \quad (41)$$

$$\frac{1}{h^{\alpha}}\sum_{j=0}^{N-M} w_j^{(\alpha)}u(hM + jh) = cx(Mh) + du(Mh) \quad (42)$$

and

$$x(0) = x_0, \qquad u(1) = 0. \quad (43)$$

These are linear equations which can be solved using a standard solver. Note that for large number of subdomains, the dimension of the resulting problem would also be large. In this case, one can use a "*short memory*" principle to reduce the computational cost. However, for the size of the problem considered, this is not a major issue. For this reason, we use the "*absolute memory*" approach.

In the following, we present some simulation results for FOC of the 2-dimensional distributed system. For simulation purposes, we take the following data: $Q' = R' = L = 1$, and the following initial conditions

$$x_0(\xi, \eta) = 1 + \beta\xi + \gamma\eta. \quad (44)$$

Using (21), we get

$$x_{ij}(0) = \begin{cases} 1 + L, & i = 0,\ j = 0 \\ \frac{2L}{(j\pi)^2}(\cos(j\pi) - 1), & i = 0,\ j = 1, 3, 5, \ldots, m \\ \frac{2L}{(i\pi)^2}(\cos(i\pi) - 1), & j = 0,\ i = 1, 3, 5, \ldots, m \\ 0, & \text{otherwise.} \end{cases} \quad (45)$$

We further take $\beta = 1$, $\gamma = 1$, and only the first few values of i and j. Once $x_{ij}(t)$ and $u_{ij}(t)$ are known, $x(\xi, \eta, t)$ and $u(\xi, \eta, t)$ can be found using (16) and (17). Simulations are performed for different values of α and N. Some of the results of the simulations are discussed below.

Figures 1 and 2 show the analytical results (for $\alpha = 1$) and the numerical results (for $\alpha = 0.6$, 0.8, 0.95, 0.99 and 1) for the state $x_{00}(t)$ and control $u_{00}(t)$

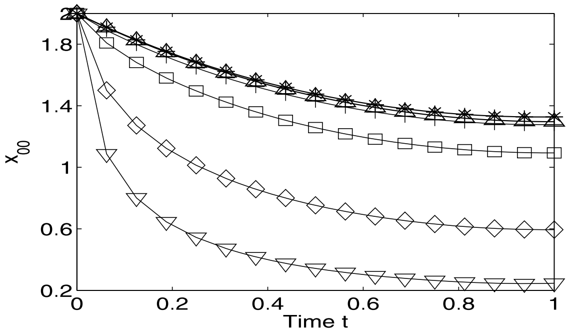

Fig. 1 Generalized eigencoordinates x_{00} as a function of time for different values of α ($\bigtriangledown : \alpha = 0.6$, $\Diamond : \alpha = 0.8$, $\Box : \alpha = 0.95$, $+ : \alpha = 0.99$. For $\alpha = 1$, $* :$ Numerical, $\triangle :$ Analytical)

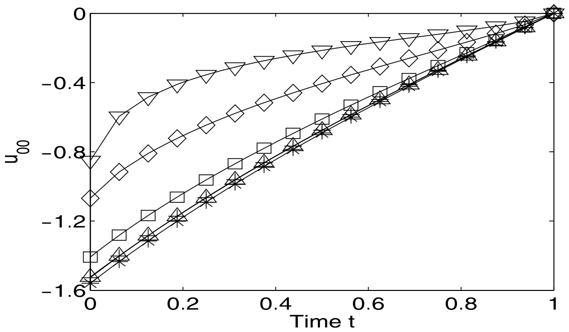

Fig. 2 Generalized eigencoordinates u_{00} as a function of time for different values of α ($\bigtriangledown : \alpha = 0.6$, $\Diamond : \alpha = 0.8$, $\Box : \alpha = 0.95$, $+ : \alpha = 0.99$. For $\alpha = 1$, $* :$ Numerical, $\triangle :$ Analytical)

eigen-coordinates as functions of time t. In this case, we take $N = 16$. Note that the analytical and numerical results for both $x_{00}(t)$ and $u_{00}(t)$ practically overlap. This indicates that as α approaches an integer value, the solution for the integer order is recovered.

Figures 3 and 4 show the state $x_{00}(t)$ and control $u_{00}(t)$ eigencoordinates as a function of time t for $\alpha = 0.75$ and different values of N. From these figures, it can be seen that the solutions converge as N is increased. However, the convergence appears to be slow. Our rough estimates suggest that the order of the scheme may be less than 1, and another numerical scheme which provides higher order of convergence may be necessary. We plan to consider this in future.

Figures 5 and 6 show that numerical results for the state $x(\xi, \eta, t)$ and control $u(\xi, \eta, t)$ as a function of time t, respectively, for $\alpha = 0.75$, $\xi = 0.25$ and $\eta = 0.25$ and $N = 20$. In this case, the indices i and j in (16) and (17) vary from 0 to 3. Following (45),

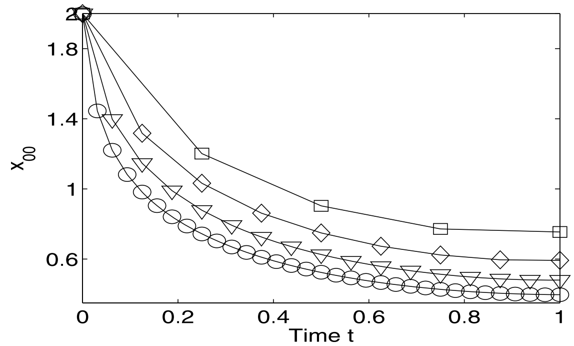

Fig. 3 Generalized eigencoordinates u_{00} as a function of time for $\alpha = 0.75$ and different N ($\alpha = 0.75$, $\Box : N = 4$, $\Diamond : N = 8$, $\bigtriangledown : N = 16$, $\bigcirc : N = 32$)

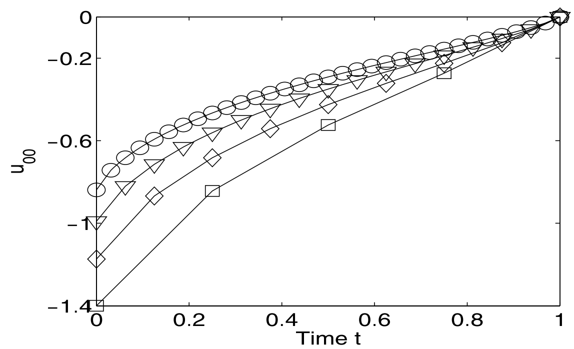

Fig. 4 Generalized eigencoordinates u_{00} as a function of time for $\alpha = 0.75$ and different N ($\alpha = 0.75$, $\Box : N = 4$, $\Diamond : N = 8$, $\bigtriangledown : N = 16$, $\bigcirc : N = 32$)

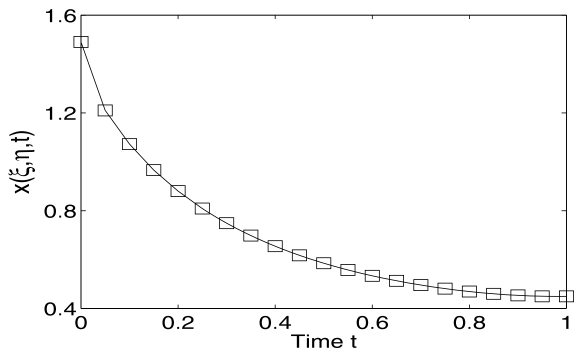

Fig. 5 $x(\xi, \eta, t)$ as a function of time for $\xi = 0.25$, $\eta = 0.25$, and $\alpha = 0.75$

for this range of i and j for (16) and the considered initial condition, we have only 5 nonzero terms and, therefore, we only need to solve 5 sets of equations.

Figures 7 and 8 show that numerical results for the state $x(\xi, \eta, t)$ and the control $u(\xi, \eta, t)$ as a function

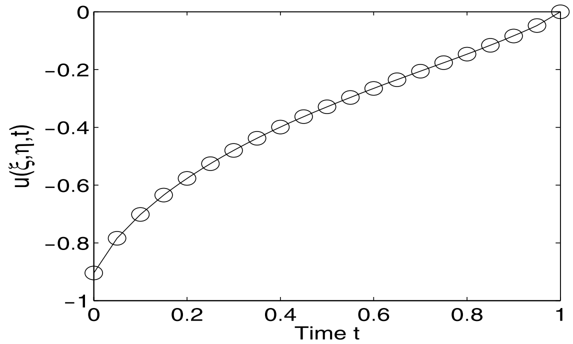

Fig. 6 $u(\xi,\eta,t)$ as a function of time for $\xi = 0.25$, $\eta = 0.25$, and $\alpha = 0.75$

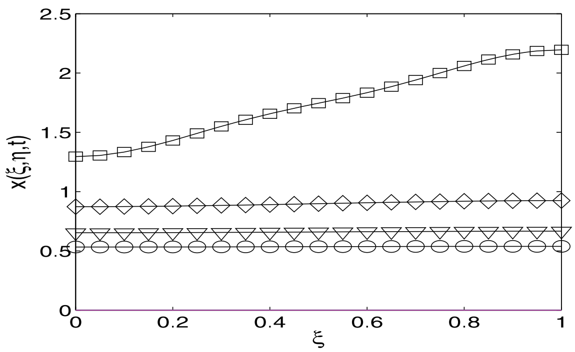

Fig. 7 $x(\xi,\eta,t)$ as a function of ξ for $\alpha = 0.75$ and ($\eta = 0.25$, $\square : t = 0$, $\lozenge : t = 0.2$, $\triangledown : t = 0.4$, $\bigcirc : t = 0.6$)

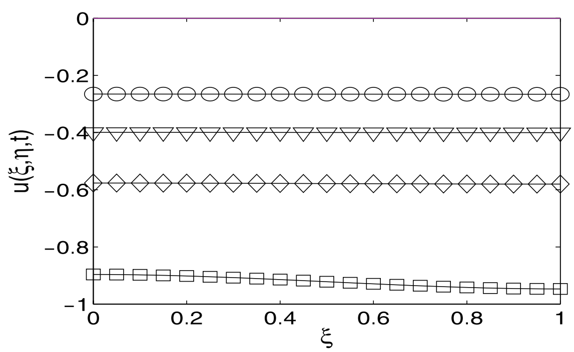

Fig. 8 $u(\xi,\eta,t)$ as a function of ξ for $\alpha = 0.75$ and ($\eta = 0.25$, $\square : t = 0$, $\lozenge : t = 0.2$, $\triangledown : t = 0.4$, $\bigcirc : t = 0.6$)

of ξ for $\alpha = 0.75$, $\eta = 0.25$, $N = 20$ and different time values $t = 0, 0.2, 0.4, 0.6$. In this case also, the indices i and j in (16) and (17) vary from 0 to 3. It is observed that only initially the state $x(\xi,\eta,t)$ and the control $u(\xi,\eta,t)$ depend on ξ, and as time t increases, they no longer depend on ξ. This is because as time progresses, the diffusion process causes the state $x(\xi,\eta,t)$ to become uniform, and as a result the control $u(\xi,\eta,t)$ also becomes uniform.

5 Conclusions

An analytical scheme for Fractional Optimal Control (FOC) of a 2-dimensional system using eigenfunctions has been presented. The fractional derivative was defined in the Riemann–Liouville sense.

The performance index of a FOC problem was considered as a function of both the state and the control variables, and the dynamic constraints were expressed by a Partial Fractional Differential Equations (PFDEs) containing two space parameters and one time parameter. Eigenfunctions were used to eliminate the terms containing space parameters, and to define the problem in terms of a set of generalized state and control variables. Grünwald–Letnikov approximation was used to approximate the fractional derivatives. A direct numerical technique was used to obtain the numerical solutions. Numerical results showed that (1) only a few eigenfunctions were sufficient to obtain good results, (2) the solutions converged as the size of the time step was reduced, and (3) in the limit as α approached to 1, the numerical results converged to analytical results.

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Appendix

In this Appendix, we present the analytical solution for $x(\xi,\eta,t)$ and $u(\xi,\eta,t)$ for $\alpha = 1$. For simplicity, the differential equations (32) and (33) and the terminal conditions (34) and (35) are rewritten as,

Differential equations:

$$\begin{cases} \dot{x}_{ij}(t) = -c_{ij}x_{ij}(t) + u_{ij}(t) \\ \dot{u}_{ij}(t) = x_{ij}(t) + c_{ij}u_{ij}(t) \end{cases} \tag{A.1}$$

$i, j = 0, 1, \ldots, m,$

The initial conditions:

$$\begin{cases} x_{ij}(0) = x_{ij0} \\ u_{ij}(1) = 0 \end{cases} \tag{A.2}$$

$i, j = 0, 1, \ldots, m,$

where

$$c_{ij} = \left(\frac{i\pi}{L}\right)^2 + \left(\frac{j\pi}{L}\right)^2, \quad i, j = 0, 1, \ldots, m, \quad \text{(A.3)}$$

and x_{ij0}, $i, j = 0, 1, \ldots, m$ are given by (21). From (A.1) we have

$$\ddot{x}_{ij}(t) - (c_{ij}^2 + 1)x_{ij}(t) = 0. \quad \text{(A.4)}$$

This is an ordinary second order differential equation for which a closed form solution can be found in a straight forward manner. Using (A.1) and (A.2), the solution of (A.4) is given as

$$\begin{aligned} x_{ij}(t) &= x_{ij0}\frac{\sqrt{c_{ij}^2+1}\cosh(\sqrt{c_{ij}^2+1}(1-t))}{\sqrt{c_{ij}^2+1}\cosh(\sqrt{c_{ij}^2+1}) + c_{ij}\sinh(\sqrt{c_{ij}^2+1})} \\ &\quad + x_{ij0}\frac{c_{ij}\sinh(\sqrt{c_{ij}^2+1}(1-t))}{\sqrt{c_{ij}^2+1}\cosh(\sqrt{c_{ij}^2+1}) + c_{ij}\sinh(\sqrt{c_{ij}^2+1})}. \end{aligned} \quad \text{(A.5)}$$

Using (A.1) and (A.5), we have

$$u_{ij}(t) = -kx_{ij0}\frac{\sinh(\sqrt{c_{ij}^2+1}(1-t))}{\sqrt{c_{ij}^2+1}\cosh(\sqrt{c_{ij}^2+1}) + c_{ij}^2\sinh(\sqrt{c_{ij}^2+1})}. \quad \text{(A.6)}$$

Finally, $x(\xi, \eta, t)$ and $u(\xi, \eta, t)$ are given by (16) and (17), where $x_{ij}(t)$ and $u_{ij}(t)$ are given by (A.5) and (A.6).

References

1. Oldham, K.B., Spanier, J.: The Fractional Calculus. Academic Press, New York (1974)
2. Miller, K.S., Ross, B.: An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley, New York (1993)
3. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives—Theory and Applications. Gordon and Breach, Longhorne (1993)
4. Carpinteri, A., Mainardi, F.: Fractal and Fractional Calculus in Continuum Mechanics. Springer, Vienna (1997)
5. Rossikhin, Y.A., Shitikova, M.V.: Applications of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanics of solids. Appl. Mech. Rev. **50**, 15–67 (1997)
6. Podlubny, I.: Fractional Differential Equations. Academic Press, New York (1999)
7. Hilfer, R.: Applications of Fractional Calculus in Physics. World Scientific, River Edge (2000)
8. Magin, R.L.: Fractional Calculus in Bioengineering. Begell House, Redding (2006)
9. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
10. Dorčak, L.: Numerical models for simulation the fractional-order control systems. UEF SAV, The Academy of Sciences Institute of Experimental Physics, Kosice, Slovak Republic (1994)
11. Dorčak, L., Lešco, V., Koštial, I.: Identification of fractional-order dynamical systems. In: Proceedings of the 12th International Conference on Process Control and Simulation ASRTP'96, pp. 62–68. Kosice, Slovak Republic (1996)
12. Podlubny, I., Dorcak, L., Kostial, I.: On fractional derivatives, fractional-order dynamic systems and $PI^{\lambda}D^{\mu}$ controllers. In: Proceedings of the 1997 36th IEEE Conference on Decision and Control, Part 5, pp. 4985–4990. San Diego, CA, USA, 10–12 December 1997
13. Vinagre, B.M., Podlubny, I., Hernández, A., Feliu, V.: Some applications of fractional order operators used in control theory and applications. Fract. Calc. Appl. Anal. **3**, 231–248 (2000)
14. Barbosa, R.S., Machado, J.A.T., Ferreira, I.M.: A Fractional calculus perspective of PID tuning. In: Proceedings of DETC03 ASME 2003 Design Engineering Technical Conference and Computers and Information in Engineering Conference Chicago, IL, 2–6 September 2003
15. Ma, C., Hori, Y.: Fractional order control and its applications of $PI^{\alpha}D$ controller robust two-inertia speed control. In: The 4th International Power Electronics and Motion Control Conference, IPEMC, 14–16 August 2004
16. Barbosa, R.S., Machado, J.A.T.: Implementation of discrete-time fractional-order controllers based on LS approximations. Acta Polytech. Hung. **3**, 4 (2006)
17. Rudas, I.J., Tar, J.K., Patkai, B.: Compensation of dynamic friction by a fractional order robust controller. In: IEEE International Conference on Computational Cybernetics, pp. 9–14. Tallinn, 20–22 August 2006
18. Outsaloup, A.: La Commande CRONE: Commande Robuste d'Ordre Non Entiére. Hermes, Paris (1991)
19. Podlubny, I.: Fractional-order systems and $PI^{\lambda}D^{\mu}$ controllers. IEEE Trans. Autom. Control **44**, 208–214 (1999)
20. Petras, I.: The fractional-order controllers: methods for their synthesis and application. J. Electr. Eng. **50**, 284–288 (1999)
21. Machado, J.A.T.: Analysis and design of fractional-order digital control systems. Syst. Anal. Model. Simul. **27**, 107–122 (1997)
22. Machado, J.A.T.: Fractional-order derivative approximations in discrete-time control systems. Syst. Anal. Model. Simul. **34**, 419–434 (1999)
23. Bagley, R.L., Calico, R.A.: Fractional order state Equations for the control viscoelastically damped structures. J. Guid. Control Dyn. **14**, 304–311 (1991)
24. Hestenes, M.R.: Calculus of Variations and Optimal Control Theory. Wiley, New York (1966)

25. Bryson, Jr., A.E., Ho, Y.C.: Applied Optimal Control: Optimization, Estimization, and Control. Blaisdell, Waltham (1975)
26. Sage, A.P., White, III, C.C.: Optimum Systems Control. Prentice-Hall, Englewood Cliffs (1977)
27. Gregory, J., Lin, C.: Constraint Optimization in the Calculus of Variations and Optimal Control Theory. Van Nostrand-Reinhold, New York (1992)
28. Agrawal, O.P.: Formulation of Euler–Lagrange equations for fractional variational problems. Math. Anal. Appl. **272**, 368–379 (2002)
29. Agrawal, O.P.: A general formulation and solution scheme for fractional optimal control problems. Nonlinear Dyn. **38**, 323–337 (2004)
30. Agrawal, O.P.: A general scheme for stochastic analysis of fractional optimal control problems. In Mahaute, A.L., Machado, J.A.T., Trigeassou, J.C., Sabatier, J. (eds.) Fractional Differentiation and Its Applications, pp. 615–624 (2005)
31. Agrawal, O.P.: A formulation and a numerical scheme for fractional optimal control problems. In: Proceedings of the 2nd IFAC Conference on Fractional Differentiations and Its Applications, IFAC (2006)
32. Agrawal, O.P.: A quadratic numerical scheme for fractional optimal control problems. J. Dyn. Syst. Measur. Control (2008, to appear)
33. Agrawal, O.P., Baleanu, D.: A Hamiltonian formulation and a direct numerical scheme for fractional optimal control problems. J. Vib. Control **13**, 1269–1281 (2007)
34. Agrawal, O.P.: Fractional Optimal Control of a Distributed System Using Eigenfunction. In: Proceedings of the 2007 ASME International Design Engineering Technical Conferences, Las Vegas, NV, 4–7 September 2007
35. Hanyga, A.: Multi-dimensional solutions of space-time-fractional diffusion euations. Proc. R. Soc. Lond. A **458**, 429–450 (2002)
36. Hanyga, A.: Multi-dimensional solutions of time-fractional diffusion-wave equations. Proc. R. Soc. Lond. A **458**, 933–957 (2002)
37. Butzer, P.L., Westphal, U.: An introduction to fractional calculus. In: Hilfer, R. (ed.) Applications of Fractional Calculus in Physics, pp. 1–85. World Scientific, Singapore (2000)