

Chapter 12

A Fractional Order Dynamical Trajectory Approach for Optimization Problem with HPM

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1 Introduction

Optimization theory is aimed to find out the optimal solution of problems which are defined mathematically from a model that arise in wide range of scientific and engineering disciplines. Many methods and algorithms have been developed for this purpose since the late 1940s. The penalty function methods are classical methods for solving nonlinear programming (NLP) problem by transforming it to the unconstrained problem, see Luenberger [1] and Sun [2] for details. Furthermore, dynamical trajectory approaches based on differential equations system are alternative methods for NLP problems. In this type of methods an optimization problem is formulated as a system of ordinary differential equations (ODEs) so that the equilibrium point of this system converges to the local minimum of the optimization problem. The methods based on ODEs for solving optimization problems have been first proposed by Arrow and Hurwicz [3] and then improved by Rosen [4], Fiacco and McCormick [5], and Yamashita [6]. Recently, Wang et al. [7], Jin et al. [8] and Özdemir and Evirgen [9, 10] have made studies in differential equation approach for solving optimization problems.

In last decade, fractional calculus has drawn a wide attention from many physicists and mathematicians, because of its interdisciplinary application and physical meaning, e.g. [11–13]. Fractional calculus deals with the generalization of differentiation and integration of noninteger order. Several analytical and numerical methods have been proposed for solving fractional differential equations (FDEs). Some commonly used techniques are summarized as follows. The variational iteration method (VIM) was first introduced by He [14], and applied to FDEs [15]. The Adomian decomposition method (ADM) [16, 17] is applied to various problems. Also, the homotopy perturbation method (HPM) is an another successful analytical

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approximate technique, which provides a solution to linear and nonlinear problems, see [18, 19]. The HPM yields a very rapid convergent series solution, and usually a few iterations lead to very accurate approximation of the exact solution [18–23]. The reason of this success is mainly based on combination of the traditional perturbation method and homotopy techniques. The HPM is used to solve a wide range of differential equations in the literature. Abdulaziz et al. [24] used HPM for solving system of FDEs. Momani and Odibat presented HPM for fractional order partial differential equation [25] and fractional quadratic Riccati differential equation was described in Odibat and Momani [26]. Baleanu et al. have solved linear and nonlinear Schrodinger equations by HPM [27]. Chowdhury and Hashim [28] have employed HPM for solving Klein Gordon equation. Furthermore, some techniques are adapted to the HPM for getting the essential behavior of the differential equation system for large time t , such as multistage and Padé approximants. The adaptation of HPM with multistage strategy for numerical and analytical solution of a system of ODEs was introduced by Hashim and Chowdhury [29]. Applications of multistage HPM for solving chaotic systems and biochemical reaction model were illustrated in [30–32], respectively.

This paper constructs a system of FDEs which is proposed to solve NLP problem with equality constraints. In order to see the coincidence between the steady state solution of the system of FDEs and the optimal solution of the NLP problem in a long time t period, we used the multistage strategy.

The paper is organized as follows. In Sect. 2, the fundamentals of optimization problem, fractional calculus and HPM are briefly reviewed. In Sect. 3, the multistage HPM is adapted to the nonlinear system of FDEs for solving NLP problem. In Sect. 4, the applicability and efficiency of multistage HPM is illustrated by comparison among traditional HPM and fourth order Runge–Kutta (RK4) method on some numerical examples. And finally some concluding remarks are given in Sect. 5.

2 Preliminaries

2.1 Optimization Problem

Consider the NLP problem with equality constraints defined by

$$\begin{aligned} & \text{minimize } f(x), \\ & \text{subject to } x \in M \end{aligned} \tag{12.1}$$

with

$$M = \{x \in \mathbb{R}^n \mid h(x) = 0\},$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and $h = (h_1, h_2, \dots, h_p)^T : \mathbb{R}^n \rightarrow \mathbb{R}^p$ ($p \leq n$). It is assumed that the functions in problem are at least twice continuously differentiable, that a solution exists, and which $\nabla h(x)$ has full rank. To obtain a solution of (12.1), the penalty function method solves a sequence of unconstrained optimization problems. A well-known penalty function for this problem is given by

$$F(x, \mu) = f(x) + \mu \frac{1}{\gamma} \sum_{l=1}^p (h_l(x))^\gamma, \quad l = 1, 2, \dots, p, \quad (12.2)$$

where $\gamma > 0$ is constant and $\mu > 0$ is an auxiliary penalty variable. The corresponding unconstrained optimization problem of (12.1) is defined as follows:

$$\min F(x, \mu) \quad \text{subject to} \quad x \in \mathbb{R}^n. \quad (12.3)$$

Further information about NLP problem can be found in Luenberger [1] and Sun [2].

2.2 Fractional Calculus

Now we will give some definitions and properties of the fractional calculus [11–13]. We begin with the Riemann–Liouville definition of the fractional integral of order $\alpha > 0$, which is given as

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \quad x > 0,$$

where $\Gamma(\cdot)$ is the Gamma function.

Most commonly encountered fractional derivatives are Riemann–Liouville and Caputo fractional derivative. The definitions of these two derivatives are given as:
Riemann–Liouville fractional derivative (RLFD)

$$D^\alpha f(x) = D^m (I^{m-\alpha} f(x)) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dt} \right)^m \int_0^x (x-t)^{m-\alpha-1} f(t) dt,$$

Caputo fractional derivative (CFD)

$${}^C D^\alpha f(x) = I^{m-\alpha} (D^m f(x)) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} \left(\frac{d}{dt} \right)^m f(t) dt,$$

where $m - 1 < \alpha \leq m$ and $m \in \mathbb{N}$. Note that D^m is the usual integer differential operator of order m . Furthermore,

$$I^\alpha D^\alpha f(x) = f(x) - \sum_{s=0}^{m-1} f^{(s)}(0^+) \frac{x^s}{s!}, \quad m - 1 < \alpha \leq m, \quad (12.4)$$

is satisfied.

2.3 Homotopy Perturbation Method

The brief outline of HPM is given in general by He in [18, 20]. For convenience, consider the following nonlinear differential equation

$$L(u) + N(u) = f(r), \quad r \in \Omega, \quad (12.5)$$

with boundary condition

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma,$$

where L is a linear operator, while N is nonlinear operator, B is a boundary operator, Γ is the boundary of the domain Ω and $f(r)$ is a known analytic function. The He's homotopy perturbation technique defines the homotopy $v(r, p) : \Omega \times [0, 1] \rightarrow \mathbb{R}$ which satisfies

$$H(v, p) = (1 - p)[L(v) - L(u_0)] + p[L(v) - N(v) - f(r)] = 0, \quad (12.6)$$

where $p \in [0, 1]$ is an embedding parameter, u_0 is an initial approximation which satisfies the boundary conditions. The changing process of p from zero to unity is just that of $v(r, p)$ from u_0 to $u(r)$. The basic assumption is that the solution of (12.6) can be expressed as a power series in p :

$$v = v_0 + pv_1 + p^2v_2 + \dots$$

The approximate solution of nonlinear equation (12.5), therefore can be readily obtained:

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots \quad (12.7)$$

The convergence of the series (12.7) has been proved in [19, 21] and the asymptotic behavior of the series is given in [22, 23].

2.4 The Runge–Kutta Method

The Runge–Kutta method is one of the well known numerical methods for differential equations. The fourth order Runge–Kutta method computes the approximate solutions of the problem $\dot{x} = f(t, x)$ by the following iterative equations:

$$\begin{aligned}x_{n+1} &= x_n + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4), \\k_1 &= f(t_n, x_n), \\k_2 &= f\left(t_n + \frac{1}{2}h, x_n + \frac{1}{2}hk_1\right), \\k_3 &= f\left(t_n + \frac{1}{2}h, x_n + \frac{1}{2}hk_2\right), \\k_4 &= f(t_n + h, x_n + hk_3),\end{aligned}$$

where h is the fixed step size $t_i - t_{i-1}$ and x_n is the estimated value of the solution at the time t_n .

3 Multistage HPM for System of FDEs

In this section we solve NLP problems which are governed by system of fractional differential equation. Consider the unconstrained optimization problem (12.3), an approach based on fractional dynamic system can be described by the following FDEs

$${}^C D^\alpha x(t) = -\nabla_x F(x, \mu), \quad (12.8)$$

subjected to the initial conditions

$$x(t_0) = x(0),$$

where ${}^C D^\alpha$ is the fractional derivative in Caputo sense of x of order α ($0 < \alpha \leq 1$).

Note that, a point x_e is called an equilibrium point of (12.8) if it satisfies the right hand side of (12.8). For convenience of reader, we reformulate fractional dynamic system (12.8) as follows:

$${}^C D^\alpha x_i(t) = g_i(t, \mu, x_1, x_2, \dots, x_n), \quad i = 1, 2, \dots, n. \quad (12.9)$$

The steady state solution of the nonlinear system of FDEs (12.9) must be coincided with local optimal solution of the NLP problem (12.1).

In order to find the solution of system (12.9), we use multistage HPM. Because the multistage strategy is provided to reach steady state solution in whole time

horizon rather than traditional HPM. According to (12.6), we have constructed the following homotopy:

$${}^C D^\alpha x_i(t) = p g_i(t, \mu, x_1, x_2, \dots, x_n), \tag{12.10}$$

where $i = 1, 2, \dots, n$ and $p \in [0, 1]$. If $p = 0$, (12.10) becomes the linear equation

$${}^C D^\alpha x_i(t) = 0,$$

and when $p = 1$, the homotopy (12.10) turns out to be the original system given in (12.9).

We assume that the system (12.9) is defined on the time interval $t \in [0, T]$. We divide the time interval into N equal length subintervals $\Delta T = T_j - T_{j-1}$, $j = 1, 2, \dots, N$ with $T_0 = 0$ and $T_N = T$. Using the parameter p , we expand the solution x_i in the following form:

$$x_i(t) = x_{i,0}(t) + p x_{i,1}(t) + p^2 x_{i,2}(t) + \dots, \quad i = 1, 2, 3, \dots, n. \tag{12.11}$$

Also, we take the initial approximations as below

$$x_{1,0}(t) = x_1(t^*), \quad x_{2,0}(t) = x_2(t^*), \dots, x_{n,0}(t) = x_n(t^*), \tag{12.12}$$

where t^* is the left end point of each subinterval and initial conditions as

$$\begin{aligned} x_{1,1}(t^*) = 0, \quad x_{2,1}(t^*) = 0, \dots, x_{n,1}(t^*) = 0 \\ \vdots \\ x_{1,K}(t^*) = 0, \quad x_{2,K}(t^*) = 0, \dots, x_{n,K}(t^*) = 0 \\ \vdots \end{aligned}$$

Substituting (12.11) into (12.10), and equating the coefficient of the terms with identical power of p , we get

$$\begin{aligned} p^0 : {}^C D^\alpha x_{i,0}(t) &= 0 \\ p^1 : {}^C D^\alpha x_{i,1}(t) &= g_{i,1}(t, \mu, x_{1,0}, \dots, x_{n,0}) \\ &\vdots \\ p^K : {}^C D^\alpha x_{i,K}(t) &= g_{i,K}(t, \mu, x_{1,0}, \dots, x_{n,0}; x_{1,1}, \dots, x_{n,1}; \dots; x_{1,K-1}, \dots, x_{n,K-1}) \\ &\vdots \end{aligned} \tag{12.13}$$

where $i = 1, 2, 3, \dots, n$ and the function $g_{i,1}, g_{i,2}, \dots$ satisfy the following equation:

$$\begin{aligned} g_i(t, \mu, x_{1,0} + p x_{1,1} + \dots, x_{2,0} + p x_{2,1} + \dots, x_{n,0} + p x_{n,1} + \dots) \\ = g_{i,1}(t, \mu, x_{1,0}, \dots, x_{n,0}) + p g_{i,2}(t, \mu, x_{1,0}, \dots, x_{n,0}; x_{1,1}, \dots, x_{n,1}) \\ + p^2 g_{i,3}(t, \mu, x_{1,0}, \dots, x_{n,0}; x_{1,1}, \dots, x_{n,1}; x_{1,2}, \dots, x_{n,2}) + \dots \end{aligned}$$

For solving the linear system (12.13), we apply the inverse operator I^α both side of equations. Therefore the components $x_{i,k}$ ($i = 1, 2, \dots, n$; $k = 0, 1, 2, \dots$) of the multistage HPM can be determined. In order to carry out the iterations for every subinterval, we have to clarify initial approximations (12.12). For this purpose we set $t^* = t_0$. In multistage HPM, the iterations provide appropriate value of solutions by means of the previous K -term approximations $\Phi_{i,K}$ of the preceding subinterval. Consequently, the approximation solution of (12.9) can be denoted as follows:

$$x_i(t) = \Phi_{i,K} = \sum_{k=0}^{K-1} x_{i,k}, \quad 1 \leq i \leq n. \quad (12.14)$$

Here the effectiveness and the applicability of the approach especially depend on choosing ΔT and the number of term in approximate solution (12.14).

4 Numerical Implementation

To illustrate the effectiveness of the multistage HPM according to the HPM and fourth order Runge–Kutta method, some test problems are taken from Hock and Schittkowski [33, 34]. Methods are coded in Maple and digits of the variables are set to 15 in all the calculations done in this paper.

Example 12.1. Consider the following NLP problem [34, Problem No: 216]

$$\begin{aligned} &\text{minimize } f(x) = 100(x_1^2 - x_2)^2 + (x_1 - 1)^2, \\ &\text{subject to } h(x) = x_1(x_1 - 4) - 2x_2 + 12 = 0. \end{aligned} \quad (12.15)$$

The optimal solution is $x^* = (2, 4)^T$. For solving the above problem, we convert it to an unconstrained optimization problem with quadratic penalty function (12.2) for $\gamma = 2$, then we have

$$F(x, \mu) = 100(x_1^2 - x_2)^2 + (x_1 - 1)^2 + \frac{1}{2}\mu(x_1(x_1 - 4) - 2x_2 + 12)^2,$$

where $\mu \in \mathbb{R}^+$ is an auxiliary penalty variable. The corresponding nonlinear system of FDEs from (12.8) is defined as

$$\left. \begin{aligned} {}^C D^\alpha x_1(t) &= -400(x_1^2 - x_2)x_1 - 2(x_1 - 1) - \mu(2x_1 - 4)(x_1^2 - 4x_1 - 2x_2 + 12), \\ {}^C D^\alpha x_2(t) &= 200(x_1^2 - x_2) + 2\mu(x_1^2 - 4x_1 - 2x_2 + 12), \end{aligned} \right\} \quad (12.16)$$

where $0 < \alpha \leq 1$. The initial conditions are $x_1(0) = 0$ and $x_2(0) = 0$. Utilizing the homotopy (12.10) with auxiliary penalty variable $\mu = 800$ and step size $\Delta T = 0.00001$, the terms of the multistage HPM solutions (12.14) are acquired. In Fig. 12.1(a)–(b) we show the approximate–exact solution x_1 and x_2 of the

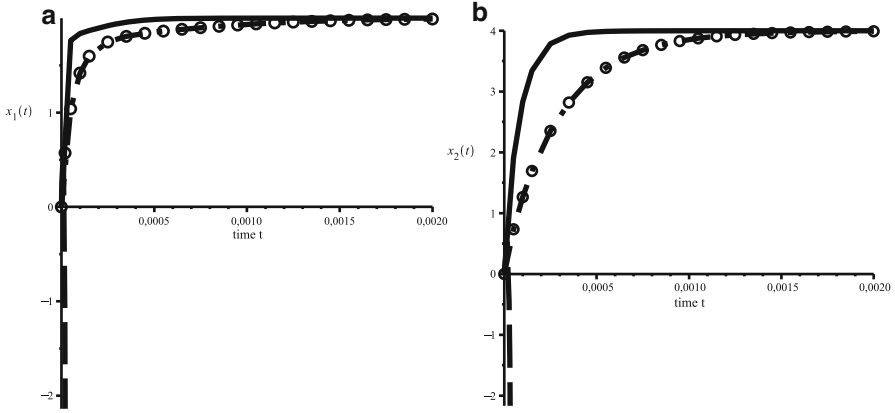


Fig. 12.1 Comparison of x_1 (a) and x_2 (b); Dash: HPM for $\alpha = 0.9$, Dashdot: MHPM ($\Delta T = 0.00001$) for $\alpha = 1$, Solidline: MHPM($\Delta T = 0.00001$) for $\alpha = 0.9$, Open circle: RK4($\Delta T = 0.00001$) for $\alpha = 1$

Table 12.1 Comparison of $x(t)$ between HPM and MHPM with RK4 solutions for different value of α

t	HPM ($\alpha = 0.9$)		MHPM ($\alpha = 0.9$)		MHPM ($\alpha = 1$)		RK4 ($\alpha = 1$)	
	$x_1(t)$	$x_2(t)$	$x_1(t)$	$x_2(t)$	$x_1(t)$	$x_2(t)$	$x_1(t)$	$x_2(t)$
0.000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.001	$-0.69E+07$	$-0.11E+07$	1.9991	3.9996	1.9338	3.8549	1.9338	3.8549
0.002	$-0.84E+08$	$-0.14E+08$	1.9993	3.9998	1.9916	3.9915	1.9916	3.9915
0.003	$-0.36E+09$	$-0.62E+08$	1.9993	3.9998	1.9986	3.9992	1.9986	3.9992
0.004	$-0.10E+10$	$-0.17E+09$	1.9993	3.9998	1.9993	3.9997	1.9992	3.9997
0.005	$-0.23E+10$	$-0.39E+09$	1.9993	3.9998	1.9994	3.9998	1.9993	3.9998

problem (12.15) for the derivative order $\alpha = 1$ and $\alpha = 0.9$. We see that for $\alpha = 1$ and $\alpha = 0.9$ our solutions obtained using the multistage HPM are in good agreement with the RK4 method solution on $x^* = (2, 4)^T$. Furthermore, the numerical results in Table 12.1 show that the multistage HPM for $\alpha = 0.9$ has better performance than for $\alpha = 1$. Clearly, the MHPM($\alpha = 0.9$) iterations converge faster than MHPM($\alpha = 1$) and RK4. So, it seems fractional order dynamical systems more realistic than integer order one for finding optimal solution of NLP problem.

Example 12.2. Consider the equality constrained optimization problem [33, Problem No: 79]

$$\begin{aligned}
 &\text{minimize } f(x) = (x_1 - 1)^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_4)^4 + (x_4 - x_5)^4 \\
 &\text{subject to } h_1(x) = x_1 + x_2^2 + x_3^3 - 2 - 3\sqrt{2} = 0, \\
 &\quad h_2(x) = x_2 - x_3^2 + x_4 + 2 - 2\sqrt{2} = 0, \\
 &\quad h_3(x) = x_1x_5 - 2 = 0.
 \end{aligned}
 \tag{12.17}$$

Table 12.2 Comparison of $x(t)$ between HPM and MHPM with RK4 solutions for different value of α

t	HPM ($\alpha = 0.9$)		MHPM ($\alpha = 0.9$)		MHPM ($\alpha = 1$)		RK4 ($\alpha = 1$)	
	$x_1(t)$	$x_2(t)$	$x_1(t)$	$x_2(t)$	$x_1(t)$	$x_2(t)$	$x_1(t)$	$x_2(t)$
0	2	2	2	2	2	2	2	2
2	0.160E + 8	0.620E + 8	1.198931	1.369223	1.182161	1.352495	1.191010	1.359541
10	0.288E + 9	0.112E + 10	1.191090	1.362530	1.191050	1.362499	1.191082	1.362524
15	0.594E + 9	0.230E + 10	1.191090	1.362530	1.191084	1.362498	1.191090	1.362530
20	0.100E + 10	0.388E + 10	1.191090	1.362530	1.191082	1.362472	1.191090	1.362530
30	0.209E + 10	0.811E + 10	1.191090	1.362530	1.191113	1.362541	1.191090	1.362530

Table 12.3 Comparison of $x(t)$ between HPM and MHPM with RK4 solutions for different value of α

t	HPM ($\alpha = 0.9$)		MHPM ($\alpha = 0.9$)		MHPM ($\alpha = 1$)		RK4 ($\alpha = 1$)	
	$x_3(t)$	$x_4(t)$	$x_3(t)$	$x_4(t)$	$x_3(t)$	$x_4(t)$	$x_3(t)$	$x_4(t)$
0	2	2	2	2	2	2	2	2
2	0.301E + 9	-0.420E + 7	1.468744	1.616076	1.478320	1.661326	1.474039	1.641529
10	0.546E + 10	-0.756E + 8	1.472774	1.634738	1.472792	1.634827	1.472778	1.634755
15	0.113E + 11	-0.156E + 9	1.472774	1.634738	1.472786	1.634792	1.472774	1.634738
20	0.191E + 11	-0.263E + 9	1.472774	1.634738	1.472798	1.634853	1.472774	1.634738
30	0.395E + 11	-0.550E + 9	1.472774	1.634738	1.472765	1.634750	1.472774	1.634738

The solution of (12.17) is $x^* \approx (1.191127, 1.362603, 1.472818, 1.635017, 1.679081)^T$, and this is not an exact solution. The equality constrained optimization problem (12.17) is transformed to an unconstrained optimization problem by using quadratic penalty function (12.2) for $\gamma = 2$ as follows:

$$F(x, \mu) = f(x) + \frac{1}{2}\mu \sum_{l=1}^3 (h_l(x))^2,$$

where $\mu \in \mathbb{R}^+$ is an auxiliary penalty variable.

The corresponding nonlinear system of FDEs from (12.8) is defined as

$${}^C D^\alpha x(t) = -\nabla f(x) - \mu \nabla h(x)h(x), \tag{12.18}$$

where $0 < \alpha \leq 1$. The initial condition is $x(0) = (2, 2, 2, 2, 2)^T$ that is not feasible. Using the homotopy (12.10) with auxiliary penalty variable $\mu = 75$ and step size $\Delta T = 0.0001$, the multistage HPM approximate-exact solutions (12.14) are obtained. In Tables 12.2–12.4, the comparison of the $x_i, i = 1, 2, 3, 4, 5$ solutions between the HPM for $\alpha = 0.9$, the multistage HPM for $\alpha = 0.9$ and $\alpha = 1$ with the classical RK4 method are given, respectively. Here, the solutions continuously depends on the order of fractional derivative. Furthermore, our approximate solutions using the multistage HPM are in good agreement with the RK4 method solution and the optimal solution of the optimization problem (12.17).

Table 12.4 Comparison of $x(t)$ between HPM and MHPM with RK4 solutions for different value of α

	HPM ($\alpha = 0.9$)	MHPM ($\alpha = 0.9$)	MHPM ($\alpha = 1$)	RK4 ($\alpha = 1$)
t	$x_5(t)$	$x_5(t)$	$x_5(t)$	$x_5(t)$
0	2	2	2	2
2	0.102E + 7	1.668076	1.691867	1.679209
10	0.183E + 8	1.679130	1.679187	1.679140
15	0.378E + 8	1.679130	1.679136	1.679130
20	0.637E + 8	1.679130	1.679142	1.679130
30	0.133E + 9	1.679130	1.679093	1.679130

5 Conclusions

In the present work, the HPM has been successfully used to obtain approximate analytical solutions of NLP problems. Initially, the NLP problem is reformulated by a system of FDEs. In order to see the essential behavior of the system of FDEs, the multistage strategy is adapted to the HPM. The numerical comparison among the fourth order Runge–Kutta (RK4), the multistage HPM ($\alpha = 0.9$ and $\alpha = 1$) and HPM ($\alpha = 0.9$) shows that the multistage HPM ($\alpha = 0.9$) performs rapid convergence to the optimal solutions of the optimization problems. Consequently, these results verify the efficiency of the multistage HPM as a practical tool for solving NLP problem.

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