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Complete Rewriting System for the Chinese Monoid

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Abstract

In this paper we are interested in the Chinese monoid and show that the Chinese monoid has complete rewriting system.

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1 Introduction

In combinatorial group and semigroup theory one of the fundamental questions is the solvability of the word problem which is one of the decision problems introduced by Max Dehn in 1911. In general this problem for finitely groups (or monoids) is not solvable; that is, given two words in the generators of the group (or monoid), there may be no algorithm to decide whether the words in fact represent the same element of the group (or monoid). So it is important to know which monoids (or groups) have solvable word problem.

Let $A = \{x_i : 1 \leq i \leq n\}$ be a well ordered set. The Chinese congruence is the congruence on A^* generated by T, where T consists of the following relations:

$$x_i x_j x_k = x_i x_k x_j = x_j x_i x_k \quad \text{for every} \quad i > j > k, \tag{1}$$

$$x_i x_j x_j = x_j x_i x_j, \;\; x_i x_i x_j = x_i x_j x_i \quad \text{for every} \quad i > j. \tag{2}$$

The Chinese monoid $CH(A)$ (of rank n) is the quotient monoid of the free monoid A^* by the Chinese congruence [5], i.e., $CH(A) = [A; T]$. Although it is easy to see that (1) and (2) together are equivalent to

$$x_i x_j x_k = x_i x_k x_j = x_j x_i x_k \quad \text{for every} \quad i \geq j \geq k, \tag{3}$$

we will exclusively use equations (1) and (2) instead of (3) in this paper. It is known that every element of $CH(A)$ (of rank n) has a unique expression of the form $x = y_1 y_2 \cdots y_n$, where

$$y_1 = x_1^{k_{11}}, \ \ y_2 = (x_2x_1)^{k_{21}} x_1^{k_{11}}, \ \ y_3 = (x_3x_1)^{k_{31}} (x_3x_2)^{k_{32}} x_3^{k_{33}},$$
$$\cdots, \ \ y_n = (x_nx_1)^{k_{n1}} (x_nx_2)^{k_{n2}} \cdots (x_nx_{n-1})^{k_{n(n-1)}} x_n^{k_{nn}},$$

with all exponents non-negative [2]. We call it the canonical form of the element $x \in CH(A)$. The Chinese monoid is related to the so called *plactic monoid* studied by Lascoux et. al. in [7]. Both constructions are strongly related to Young tableaux, and therefore to representation theory and algebraic combinatorics. This monoid appeared in the classification of classes monoids with the growth function coinciding with that of the plactic monoid (see [5]). Then combinatorial properties of this kind monoid were studied in detail in [2]. After that in [6], authors studied the structure of the algebra $K[M]$ of the Chinese monoid M of rank 3 over a field K. As a last work, in [3] authors simplified some part of the paper [2] by using the Gröbner-Shirshov bases theory for associative algebras. So there are no any works on the Chinese monoid with the point of geometric approaches in the literature. That is why we studied on the Chinese monoid.

In this work we focus on the Chinese monoid with rank 3 since the general meaning of this case can be considered similarly. Hence we have the Chinese monoid with rank 3 as follows

$$\begin{aligned}\mathcal{P}_{M_3} = [x_1, x_2, x_3 \quad ; \quad & x_3x_2x_1 = x_2x_3x_1, x_3x_1x_2 = x_2x_3x_1,\\ & x_2x_1x_1 = x_1x_2x_1, x_3x_2x_2 = x_2x_3x_2,\\ & x_3x_1x_1 = x_1x_3x_1, x_2x_2x_1 = x_2x_1x_2,\\ & x_3x_3x_2 = x_3x_2x_3, x_3x_3x_1 = x_3x_1x_3]\end{aligned} \tag{4}$$

where $3 > 2 > 1$ and show that the Chinese monoid has complete rewriting system.

2 Rewriting Systems

Let us first recall some fundamental material that needed in the proof. We note that the reader is referred to, for instance, [1, 9] for a detailed survey on (complete) rewriting sytems.

Let X be a set and let X^* be the free monoid consists of all words obtained by the elements of X. A (string) *rewriting system* on X^* is a subset $R \subseteq X^* \times X^*$ and an element $(u, v) \in R$, also written $u \to v$, is called a rule of R. The idea for a rewriting system is an algorithm for substituting the right-hand side of a rule whenever the left-hand side appears in a word. In general, for a given rewriting system R, we write $x \to y$ for $x, y \in X^*$ if $x = uv_1w$, $y = uv_2w$ and $(v_1, v_2) \in R$. Also we write $x \to^* y$ if $x = y$ or $x \to x_1 \to x_2 \to \cdots \to y$ for some finite chain of reductions and $\leftrightarrow^*$ is the reflexive, symmetric, and transitive closure of $\to$. Furthermore an element $x \in X^*$ is called irreducible with respect to R if there is no possible rewriting (or reduction) $x \to y$; otherwise x is called *reducible*. The rewriting system R is called

- *Noetherian* if there is no infinite chain of rewritings $x \to x_1 \to x_2 \to \cdots$ for any word $x \in X^*$,
- *Weight-reducing* if there exists a weight function $g : X \to \mathbb{N}_+$ such that the extension of g to a morphism $g : X^* \to \mathbb{N}$ satisfies $g(u) > g(v)$ for each rule $u \to v \in R$,
- *Confluent* if whenever $x \to^* y_1$ and $x \to^* y_2$, there is a $z \in X^*$ such that $y_1 \to^* z$ and $y_2 \to^* z$,
- *Complete* if R is both Noetherian and confluent.

A rewriting system is *finite* if both X and R are finite sets. Besides it is undecidable in general whether or not a given finite rewriting system is Noetherian. On the other hand, if $>$ is an admissible well-founded partial ordering on X^* such that R is compatible with $>$, that is $u > v$ holds for each rule $u \to v \in R$, then R is necessarily Noetherian. Furthermore a *critical pair* of a rewriting system R is a pair of overlapping rules if one of the

(i) (r_1r_2, s), $(r_2r_3, t) \in R$ with $r_2 \neq 1$ or **(ii)** $(r_1r_2r_3, s)$ $(r_2, t) \in R$,

form is satisfied. Also a critical pair is *resolved* in R if there is a word z such that $sr_3 \to^* z$ and $r_1t \to^* z$ in the first case or $s \to^* z$ and $r_1tr_3 \to^* z$ in the second. A Noetherian rewriting system is complete if and only if every critical pair is resolved ([9]). Knuth and Bendix have developed an *algorithm* for creating a complete rewriting system R' which is equivalent to R, so that any word over X has an (unique) irreducible form with respect to R'. By considering overlaps of left-hand sides of rules, this algorithm basicly proceeds forming new rules when two reductions of an overlap word result in two distinct reduced forms.

By considering the rewrite rules in (4) we have the following result.

Lemma 2.1 *The rewriting system for the Chinese monoid (on three generators) is complete.*

Proof. To show that this system is Noetherian, we have a weight function $\omega_0 : X \to \mathbb{N}^+$ ($X = \{x_1, x_2, x_3\}$) such that the extension of ω_0 to a morphism $\omega : X^* \to \mathbb{N}$ as $\omega(u) := i_1 2^n + i_2 2^{n-1} + \cdots + i_{n-1} 2^2 + i_n 2$ for any $u = x_{i_1} x_{i_2} \cdots x_{i_n}$, $i_j \in \{1, 2, 3\}$ satisfies $\omega(u_r) > \omega(v_r)$ for each rule $u_r \to v_r$ ($1 \leq r \leq 8$) in (4). Now let us consider overlap words $x_3x_2x_1x_1$, $x_3x_1x_2x_1x_1$, $x_3x_1x_2x_2x_1$, $x_2x_2x_1x_1$, $x_3x_2x_2x_1x_1$, $x_3x_2x_2x_1$, $x_3x_3x_2x_1x_1$, $x_3x_3x_2x_2x_1$, $x_3x_3x_2x_1$, $x_3x_3x_1x_2$, $x_3x_2x_2x_2x_1$. Then we see that all overlap words are resolved by Figure 1 which is based on reduction steps.

In Figure 1, there is one word that has two different irreducible words getting by left and right reduction steps. So to have a complete system, we need to apply classical Knuth-Bendix algorithm. In this way we need to add the rule $x_3x_2x_3x_1 = x_3x_1x_3x_2$ to the rule set and check all overlap words again. By checking these overlap words from left and right side by reduction steps, we obtain one irreducible words for each overlap words. This gives us having complete presentation for the Chinese monoid. $\diamondsuit$

We recall that for a rewriting system R over X, the word problem is the following decision problem:

Instance: Two strings $u, v \in X^*$.

Question: Does $u \leftrightarrow^* v$ hold?

Lemma 2.1 shows that the Chinese monoid (on three generators) has solvable word problem since all words have a unique reduced word. Now for any number of generators of the Chinese monoid, we let

$$\begin{aligned} &1.\ x_ix_jx_k \to x_jx_ix_k, \qquad 2.\ x_ix_kx_j \to x_jx_ix_k, \qquad 3.\ x_ix_jx_j \to x_jx_ix_j, \\ &4.\ x_ix_ix_j \to x_ix_jx_i, \qquad 5.\ x_ix_jx_ix_k \to x_ix_kx_ix_j. \end{aligned} \tag{5}$$

For generators x_i, x_j, x_k, we have the rule 5. as a generalization of the rule $x_3x_2x_3x_1 \to x_3x_1x_3x_2$ which is obtained by Knuth-Bendix algorithm (see [1], [9]). These rules given in (5) form a complete rewriting system for the Chinese

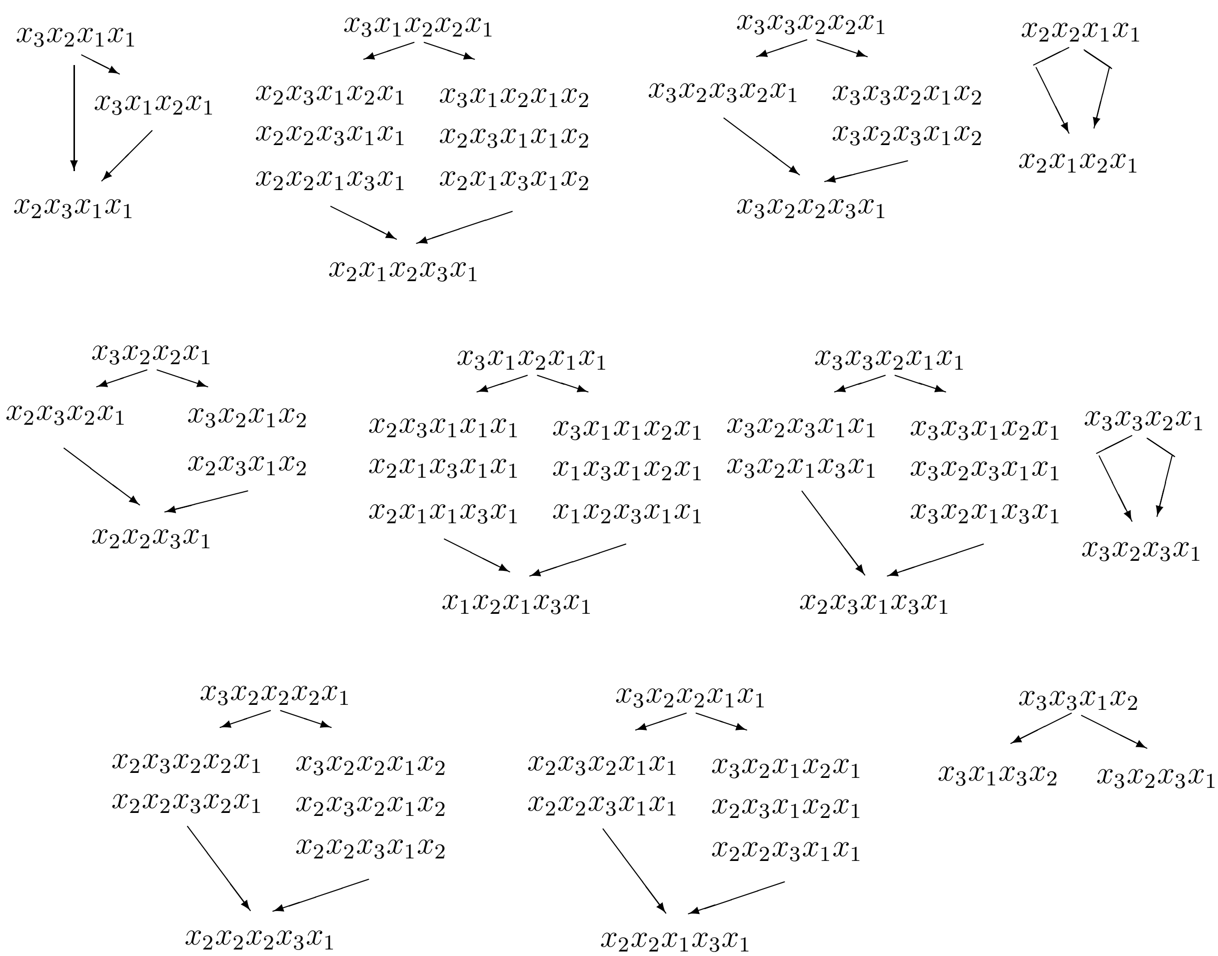

Figure 1:

monoid. To see that, we need to check all overlap words given in Table 1. (Here overlap of one rule, $p.$, by another one, $q.$, is denoted by $p : q$, e.g. overlap of 1. by 2. is $1 : 2$. Besides that the third column shows coincide generators in an overlap word).

Theorem 2.2 *The Chinese monoid given by relations* (1) *and* (2) *has complete rewriting system.*

Proof. By Lemma 2.1 and considering all overlap words in Table 1 it is seen that the rewriting system (5) is complete. Hence the Chinese monoid given by relations (1) and (2) has complete rewriting system. $\diamondsuit$

By Theorem 2.2, there is an algorithm to find the normal form for each element in A^*. So we get the following result.

Table 1: Overlap words for the Chinese monoid

p:q	*overlap word*		*ordering*	*p:q*	*overlap word*		*ordering*
1 : 1	$x_ix_jx_kx_q$	$\{x_j, x_k\}$	$i>j>k>q$	4 : 3	$x_ix_ix_jx_j$	$\{x_i, x_j\}$	$i>j$
1 : 1	$x_ix_jx_kx_px_q$	$\{x_k\}$	$i>j>k>p>q$	4 : 3	$x_ix_ix_jx_px_p$	$\{x_j\}$	$i>j>p$
1 : 2	$x_ix_jx_px_k$	$\{x_j, x_p\}$	$i>j>p>k$	4 : 4	$x_ix_ix_jx_jx_p$	$\{x_j\}$	$i>j>p$
1 : 2	$x_ix_jx_kx_px_q$	$\{x_k\}$	$i>j>k>p>q$	1 : 5	$x_ix_jx_kx_jx_q$	$\{x_j, x_k\}$	$i>j>k>q$
1 : 3	$x_ix_jx_kx_k$	$\{x_j, x_k\}$	$i>j>k$	1 : 5	$x_ix_jx_kx_px_kx_q$	$\{x_k\}$	$i>j>k>p>q$
1 : 3	$x_ix_jx_kx_px_p$	$\{x_k\}$	$i>j>k>p$	2 : 5	$x_ix_kx_jx_px_jx_q$	$\{x_j\}$	$i>j>k, j>p>q$
1 : 4	$x_ix_jx_kx_kx_p$	$\{x_k\}$	$i>j>k>p$	3 : 5	$x_ix_jx_jx_px_jx_q$	$\{x_j\}$	$i>j>p>q$
2 : 1	$x_ix_kx_jx_px_q$	$\{x_j\}$	$i>j>k, j>p>q$	4 : 5	$x_ix_ix_jx_ix_p$	$\{x_i, x_j\}$	$i>j>p$
2 : 2	$x_ix_kx_jx_qx_p$	$\{x_j\}$	$i>j>k, j>p>q$	4 : 5	$x_ix_ix_jx_qx_jx_p$	$\{x_j\}$	$i>j>p>q$
2 : 3	$x_ix_kx_jx_px_p$	$\{x_j\}$	$i>j>k, j>p$	5 : 1	$x_ix_jx_ix_kx_q$	$\{x_i, x_k\}$	$i>j>k>q$
2 : 4	$x_ix_kx_jx_jx_p$	$\{x_j\}$	$i>j>k, j>p$	5 : 1	$x_ix_jx_ix_kx_px_q$	$\{x_k\}$	$i>j>k>p>q$
3 : 1	$x_ix_jx_jx_px_q$	$\{x_j\}$	$i>j>p>q$	5 : 2	$x_ix_jx_ix_kx_p$	$\{x_i, x_k\}$	$i>j>k, i>p>k$
3 : 2	$x_ix_jx_jx_qx_p$	$\{x_j\}$	$i>j>p>q$	5 : 2	$x_ix_jx_ix_kx_qx_p$	$\{x_k\}$	$i>j>k>p>q$
3 : 3	$x_ix_jx_jx_px_p$	$\{x_j\}$	$i>j>p$	5 : 3	$x_ix_jx_ix_kx_k$	$\{x_i, x_k\}$	$i>j>k$
3 : 4	$x_ix_jx_jx_p$	$\{x_j, x_j\}$	$i>j>p$	5 : 3	$x_ix_jx_ix_kx_px_p$	$\{x_k\}$	$i>j>k>p$
4 : 1	$x_ix_ix_jx_k$	$\{x_i, x_j\}$	$i>j>k$	5 : 4	$x_ix_jx_ix_kx_kx_p$	$\{x_k\}$	$i>j>k>p$
4 : 1	$x_ix_ix_jx_px_q$	$\{x_j\}$	$i>j>p>q$	5 : 5	$x_ix_jx_ix_kx_ix_q$	$\{x_i, x_k\}$	$i>j>k>q$
4 : 2	$x_ix_ix_jx_p$	$\{x_i, x_j\}$	$i>p>j$	5 : 5	$x_ix_jx_ix_kx_px_kx_q$	$\{x_k\}$	$i>j>k>p>q$
4 : 2	$x_ix_ix_jx_qx_p$	$\{x_j\}$	$i>j>p>q$				

Corollary 2.3 *The word problem is solvable for the Chinese monoid.*

In fact the complete rewriting system for the Chinese monoid obtained in this paper as formed in (5) coincides with Gröbner-Shirshov bases for the same monoid defined in [3]. This shows that this paper is important for revealed this important fact.

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