# The isomorphism between two fundamental groups by Cayley graphs

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# The isomorphism between two fundamental groups by Cayley graphs

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**Abstract**

Let $G_1$ and $G_2$ be two finite groups and let $Cay(G_1, S_1)$ and $Cay(G_2, S_2)$ be the corresponding Cayley graphs of these groups, respectively. By [2] and [8], one can define the fundamental group $\pi_1(\Gamma, v)$ by using any connected graph $\Gamma$ with a fixed vertex $v$. In this paper we give sufficient conditions for any two fundamental groups which are obtained by Cayley graphs $Cay(G_1, S_1)$ and $Cay(G_2, S_2)$ to be isomorphic. At the final part of the paper, we present some examples of this result.

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## 1 Introduction

In this section we recover some basic material about the Cayley graphs and the fundamental groups.

Let $G$ be a finite group, and let $S$ be a generating set of $G$. Let $V(G, S)$ be the set of vertices and let $E(G, S)$ be the set of edges defined by

$$\begin{array}{rcl} V(G,S) & : & \text{The elements of } G, \\ E(G,S) & : & \text{The elements of the set } G \times S = \{(g,s) : g \in G, s \in S\} \\ & & \text{and their inverses.} \end{array}$$

Then the graph obtained by the above sets is called *Cayley graph* of $G$ and denoted by $Cay(G, S)$. The initial vertex of the edge $(g, s)$ is $g$ and the terminal is $gs$. Also the inverse of the edge $(g, s)$ is given by $(gs, s^{-1})$. In other words,

$$\iota(g,s) = g, \quad \tau(g,s) = gs \ \text{ and } \ (g,s)^{-1} = (gs, s^{-1}).$$

Therefore the equalities

$$\iota(g,s) = \tau((g,s)^{-1}) \quad \tau(g,s) = \iota((g,s)^{-1}) \ \text{ and } \ ((g,s)^{-1})^{-1} = (g,s)$$

are hold. Since the direction of the edges are different than each other, we have $(g, s) \neq (g, s)^{-1}$. These above material give us that $Cay(G, S)$ is actually defined as a graph. Similar definitions for Cayley graphs can also be found in [1], [3], [6], [7].

To state our main result, we need to recall some basic facts about the fundamental groups as well. We note that the reader can find the details of the following facts, for instance, in [2], [8].

Let $\Gamma$ be a graph and let $V$, $E$ be the vertex and edge sets of $\Gamma$, respectively. A *path* $\alpha$ is a sequence of edges $e_1 e_2 \cdots e_n$ where $\tau(e_i) = \iota(e_{i+1})$, for $i = 1, 2, \cdots, n-1$ and $e_i \in E$. An elementary operation of a path is the elimination (or insertion) of a pair ${e_i}^{\varepsilon} {e_i}^{-\varepsilon}$ in this path. Two paths $\alpha, \alpha^*$ are equivalent if there are paths

$$\alpha = \alpha_0, \alpha_1, \cdots, \alpha_n = \alpha^*$$

such that $\alpha_{i+1}$ is obtained from $\alpha_i$ by an elemantary elimination (or insertion), for $i = 1, 2, \cdots, n-1$. We then write $\alpha \sim \alpha^*$ and denote the equivalence class of $\alpha$ by $[\alpha]$.

If $\alpha$, $\beta$ are paths in $\Gamma$ then we say that the product $\alpha\beta$ is defined if $\tau(\alpha) = \iota(\beta)$. In this case $\alpha\beta$ is the path consisting of edges of $\alpha$ followed by the edges of $\beta$ (so this product is called *partial multiplication*). Then it is easy to show that if $\alpha \sim \alpha^*$, $\beta \sim \beta^*$ then $\alpha\beta$ is defined and so $\alpha^*\beta^*$. Also $\alpha \sim \alpha^*$ and $\beta \sim \beta^*$ gives that $\alpha\beta \sim \alpha^*\beta^*$.

We define a partial multiplication of equivalence classes by

$$[\alpha][\beta] = [\alpha\beta] \quad \text{where} \quad \tau(\alpha) = \iota(\beta) \tag{1}$$

which is well-defined by the above paragraph. Now let us fix a vertex $v \in V$ of $\Gamma$, and consider the set

$$\{[\alpha] \ : \ \iota(\alpha) = \tau(\alpha) = v\}. \tag{2}$$

Then we can multiply any two elements of this set since $[\alpha][\beta] = [\alpha\beta]$ and $\tau(\alpha) = \iota(\beta)$.

The set (2) with the multiplication, as in (1), defines a group where the identity element is $[1_v]$ and the inverse of an element $[\alpha]$ is $[\alpha^{-1}]$. This group is called the *fundamental group* of $\Gamma$ at $v$ and denoted by $\pi_1(\Gamma, v)$.

# 2 The main theorem

Let $G_1$, $G_2$ be finite groups with the minimal number of generating sets $S_1$, $S_2$, respectively and let $Cay(G_1, S_1)$, $Cay(G_2, S_2)$ be the corresponding Cayley graphs of $G_1$ and $G_2$, respectively. Also let $v_1$ and $v_2$ be any two vertices in $Cay(G_1, S_1)$ and $Cay(G_2, S_2)$. We then have the following result as a main theorem of this paper.

**Theorem 2.1** *$\pi_1(Cay(G_1, S_1), v_1) \cong \pi_1(Cay(G_2, S_2), v_2)$ if $|S_1| = |S_2|$ and $|G_1| = |G_2|$ where $|\,.\,|$ denotes the number of elements in the set.*

# 3 Proof of the main theorem

To prove Theorem 2.1 we need to remind some results which the proofs of them can be found in [2], [4] and [8].

Let $F(\mathbf{X})$ and $F(\mathbf{Y})$ be the free groups with the generating sets $\mathbf{X}$ and $\mathbf{Y}$, respectively.

**Proposition 3.1**

$$F(\mathbf{X}) \cong F(\mathbf{Y}) \Leftrightarrow rk(\mathbf{X}) = rk(\mathbf{Y}),$$

*where $rk(\,.\,)$ denotes the rank of the set.*

Let $\Gamma$ be any graph and let $\pi_1(\Gamma, v)$ be a fundamental group defined on $\Gamma$.

**Proposition 3.2** *$\pi_1(\Gamma, v)$ is a free group.*

**Proposition 3.3** *If $u$ and $v$ are two vertices in $\Gamma$ such that $u$, $v$ can be joined by a path then $\pi_1(\Gamma, u) \cong \pi_1(\Gamma, v)$.*

Let $T$ be a maximal tree (see, for instance [9] and [11]) in $\Gamma$ and, for $v, v_1 \in V$, let $\gamma_{v_1}$ be a geodesic (that is, the smallest path from $v$ to $v_1$) in $T$. Then one can define the elements of the generating set of the fundamental group as $t_e = [\gamma_{\iota(e)} e \gamma_{\tau(e)}^{-1}]$ where $e \in E$ but $e \notin T$. It is clear that the total number of elements in the generating set gives the rank of the fundamental group. So, for a fixed $v \in V$, let us denote the rank of the fundamental group $\pi_1(\Gamma, v)$ by $rk(\mathbf{X}_{\pi_1(\Gamma,v)})$. We then have the following result.

**Theorem 3.4** *([2], [8]) Let $\Gamma$ be a connected graph and let $v$ be a vertex in $\Gamma$. Suppose that the number of edges is $2n$ and the number of vertices is $m \neq 0$ $(n, m \in \mathbb{N})$ in $\Gamma$. Then the rank of $\pi_1(\Gamma, v)$ is $n - m + 1$.*

Now by considering the Cayley graph $Cay(G, S)$ for a finite group $G$, we can prove the following lemmas.

**Lemma 3.5**

$$|V(G, S)| = |G| \quad \textit{and} \quad |E(G, S)| = 2|G||S|.$$

**Proof.** By the definition of Cayley graphs, since the vertices of $Cay(G, S)$ are the elements of $G$ then it is clear that $|V(G, S)| = |G|$. Moreover, again by the definition, the edge set of this Cayley graph is obtained by the elements of the set $G \times S$ and their inverses then the number of elements of $E(G, S)$ is equal twice the number of elements in $G \times S$. In other words $|E(G, S)| = 2|G||S|$, as required. $\diamondsuit$

**Lemma 3.6** *$Cay(G, S)$ is connected.*

**Proof.** It is well known that to see a graph is connected, it is enough to show that each vertex is combined to a fixed vertex by a path in that graph. Since $G$ is finite, let us assume that the generating set of $G$ is $S = \{s_1, s_2, \cdots, s_n\}$. Thus, for all $g \in G$, the element $g$ of $G$ can be written by $g = s_{g1}^{\varepsilon_1} s_{g2}^{\varepsilon_2} \cdots s_{gn}^{\varepsilon_n}$ ($s_{gi} \in S$, $\varepsilon_i = \pm 1$ and $1 \leq i \leq n$). Then it is easy to see that the vertex $1 \in G$ can be combined to the vertex $g$ by the path

$$\rho = (1, s_{g1}^{\varepsilon_1})(s_{g1}^{\varepsilon_1}, s_{g2}^{\varepsilon_2}) \cdots (s_{g1}^{\varepsilon_1} s_{g2}^{\varepsilon_2} \cdots s_{g(n-1)}^{\varepsilon_{n-1}}, s_{gn}^{\varepsilon_n}).$$

For the path $\rho$, we have $\iota(\rho) = 1$ and $\tau(\rho) = s_{g1}^{\varepsilon_1} s_{g2}^{\varepsilon_2} \cdots s_{gn}^{\varepsilon_n} = g$. By applying this procedure for every element of $G$, we can see that the Cayley graph $Cay(G, S)$ is connected, as required. $\diamondsuit$

Now we can prove our **main theorem** as follows.

Let us assume that $|G_1| = |G_2| = m$ and $|S_1| = |S_2| = n$ where $m, n \in \mathbb{Z}^+$. By Lemma 3.5, for the Cayley graphs $Cay(G_1, S_1)$, $Cay(G_2, S_2)$, the number of elements in the edge sets $E(G_1, S_1)$ and $E(G_2, S_2)$ is $2|G_1||S_1|$ and $2|G_2||S_2|$, respectively. Therefore, by the assumption, the number of edges are equal in these both Cayley graphs and this number is $2mn$. Also, by the definition of Cayley graphs, the number of vertices in $Cay(G_1, S_1)$ is $|G_1|$ and similarly, the number of vertices in $Cay(G_2, S_2)$ is $|G_2|$. Thus, by the assumption, the number of vertices in both Cayley graphs are equal and this number is $m$.

By Proposition 3.2, we know that the fundamental groups are free. Also, by Lemma 3.6, the Cayley graphs $Cay(G_1, S_1)$ and $Cay(G_2, S_2)$ are connected. By Proposition 3.3, since $Cay(G_1, S_1)$ is connected then, for each $u \in V(G_1, S_1)$, the fundamental groups of $Cay(G_1, S_1)$ at $u$ are isomorphic. Similarly, for each $v \in V(G_2, S_2)$, the fundamental groups of $Cay(G_2, S_2)$ at $v$ are isomorphic. In this proof, since we are checking the case of isomorphism between two fundamental groups which are free, then we must count the rank of each $\pi_1(Cay(G_1, S_1), u)$ and $\pi_1(Cay(G_2, S_2), v)$. In fact, by Theorem 3.4, the each rank of $\pi_1(Cay(G_1, S_1), u)$ and $\pi_1(Cay(G_2, S_2), v)$ is $mn - m + 1$.

Thus we have

$$rk(\mathbf{X}_{\pi_1(Cay(G_1,S_1),u)}) = rk(\mathbf{X}_{\pi_1(Cay(G_2,S_2),v)}).$$

Then, by Proposition 3.1,

$$\pi_1(Cay(G_1, S_1), u) \cong \pi_1(Cay(G_2, S_2), v),$$

as required.

Hence the result. $\diamondsuit$

**Remark 3.7** *The inverse of Theorem* 2.1 *is not always true. To see this let us assume that*

$$\pi_1(Cay(G_1, S_1), u) \cong \pi_1(Cay(G_2, S_2), v),$$

*where $u \in V(G_1, S_1)$ and $v \in V(G_2, S_2)$. Then, by Proposition 3.1, we have*

$$rk(\mathbf{X}_{\pi_1(Cay(G_1,S_1),u)}) = rk(\mathbf{X}_{\pi_1(Cay(G_2,S_2),v)}).$$

*But this equality does not imply the conditions $|G_1| = |G_2|$ and $|S_1| = |S_2|$ hold.*

# 4 Some examples

In this section we will consider some examples and applications of Theorem 2.1. We should note that the notation $\mathbb{Z}_n$ denotes the cyclic group of order $n$ at the rest of the paper.

**Example 4.1** *Let $G_1 = \mathcal{V}_4$ (Klein 4-group) and $G_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$. By [4], these groups are presented by*

$$\mathcal{P}_1 = \left\langle a, b \; ; \; a^2, b^2, (ab)^2 \right\rangle \quad and \quad \mathcal{P}_2 = \left\langle c, d \; ; \; c^2, d^2, cdc^{-1}d^{-1} \right\rangle.$$

*Then, by using these presentations, it is easy to draw the Cayley graphs corresponding these groups (see [10]). It is well known that $G_1 \cong G_2$. Then $|G_1| = |G_2| = 4$ and $|S_1| = |S_2| = 2$. Thus, by Theorem 2.1,*

$$\pi_1(Cay(G_1, S_1), u) \cong \pi_1(Cay(G_2, S_2), v),$$

*for any $u \in V(G_1, S_1)$ and $v \in V(G_2, S_2)$.*

**Example 4.2** *Let $G_1 = \mathcal{S}_3$ (permutation group) and $G_2 = \mathbb{Z}_2 \times \mathbb{Z}_3$. It is clear that these groups are not isomorphic. Again, by [4], these groups are presented by*

$$\mathcal{P}_1 = \left\langle a, b \; ; \; a^2, b^3, (ab)^2 \right\rangle \quad and \quad \mathcal{P}_2 = \left\langle c, d \; ; \; c^2, d^3, cdc^{-1}d^{-1} \right\rangle.$$

*Then one can easily draw the Cayley graphs of these groups (see [10] for the details). It is also clear that $|G_1| = |G_2| = 6$ and $|S_1| = |S_2| = 2$. Thus, by Theorem 2.1, we can get the isomorphism between two fundamental groups obtained by the Cayley graphs over these groups.*

As a consequence of Theorem 2.1 and Example 4.2, we have the following result.

**Corollary 4.3** *Let $G_1$ and $G_2$ be two finite groups. Suppose that they are not isomorphic to each other. Then $\pi_1(Cay(G_1, S_1), u) \cong \pi_1(Cay(G_2, S_2), v)$ if*

$$rk(\mathbf{X}_{\pi_1(Cay(G_1,S_1),u)}) = rk(\mathbf{X}_{\pi_1(Cay(G_2,S_2),v)}),$$

*where $u \in V(G_1, S_1)$, $v \in V(G_2, S_2)$.*

**Example 4.4** *For any $m, n \in \mathbb{Z}^+$, it is well known that $\mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn}$ if and only if $(m, n) = 1$. Also, by [4], these groups are presented by*

$$\mathcal{P}_1 = \langle a, b\ ;\ a^m, b^n, [a, b]\rangle \quad \text{and} \quad \mathcal{P}_2 = \langle c\ ;\ c^{mn}\rangle\,.$$

*As in the previous examples, by [11], one can draw the Cayley graphs of these groups. Let us denote the Cayley graph of $G_1 = \mathbb{Z}_m \times \mathbb{Z}_n$ by $Cay(G_1, S_1)$ and the Cayley graph of $G_2 = \mathbb{Z}_{mn}$ by $Cay(G_2, S_2)$. Clearly $|G_1| = |G_2| = mn$ and $|S_1| = 2$, $|S_2| = 1$.*

*For a fixed $u \in V(G_1, S_1)$, we have the fundamental group $\pi_1(Cay(G_1, S_1), u)$ and*

$$rk(\mathbf{X}_{\pi_1(Cay(G_1,S_1),u)}) = 2mn - mn + 1.$$

*Similarly, for a fixed $v \in V(G_2, S_2)$, we have the fundamental group $\pi_1(Cay(G_2, S_2), v)$ and*

$$rk(\mathbf{X}_{\pi_1(Cay(G_2,S_2),v)}) = mn - mn + 1 = 1.$$

*Therefore $rk(\mathbf{X}_{\pi_1(Cay(G_1,S_1),u)}) \neq rk(\mathbf{X}_{\pi_1(Cay(G_2,S_2),v)})$ so, by Proposition 3.1, $\pi_1(Cay(G_1, S_1), u)$ and $\pi_1(Cay(G_2, S_2), v)$ are not isomorphic.*

As a consequence of Theorem 2.1 and Example 4.4, we have the following result.

**Corollary 4.5** *For finite groups $G_1$ and $G_2$, if $G_1 \cong G_2$ such that*

$$rk(\mathbf{X}_{\pi_1(Cay(G_1,S_1),u)}) \neq rk(\mathbf{X}_{\pi_1(Cay(G_2,S_2),v)})$$

*then, for a fixed $u \in V(G_1, S_1)$ and $v \in V(G_2, S_2)$, the fundamental groups $\pi_1(Cay(G_1, S_1), u)$ and $\pi_1(Cay(G_2, S_2), v)$ are not isomorphic.*

**Questions:**
**1)** Is it possible to expand Theorem 2.1 as necessity and sufficiency?
**2)** Can the subject of *Schur multiplier* (see [5]) be a method to show that the isomorphism between two fundamental groups?

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